QUANTUM $\mathfrak{gl}_\infty$, INFINITE q-SCHUR ALGEBRAS AND THEIR REPRESENTATIONS

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Abstract. In this paper, we investigate the structure and representations of the quantum group $\mathbf{U}(\infty)=\mathbf{U}_v(\mathfrak{gl}_\infty)$. We will present a realization for $\mathbf{U}(\infty)$, following Beilinson–Lusztig–MacPherson (BLM) [1], and show that the natural algebra homomorphism ζ_r from $\mathbf{U}(\infty)$ to the infinite q-Schur algebra $\boldsymbol{\mathcal{S}}(\infty,r)$ is not surjective for any $r\geqslant 1$. We will give a BLM type realization for the image $\mathbf{U}(\infty,r):=\zeta_r(\mathbf{U}(\infty))$ and discuss its presentation in terms of generators and relations. We further construct a certain completion algebra $\widehat{\boldsymbol{\mathcal{K}}}^\dagger(\infty)$ so that ζ_r can be extended to an algebra epimorphism $\widetilde{\zeta}_r:\widehat{\boldsymbol{\mathcal{K}}}^\dagger(\infty)\to\boldsymbol{\mathcal{S}}(\infty,r)$. Finally we will investigate the representation theory of $\mathbf{U}(\infty)$, especially the polynomial representations of $\mathbf{U}(\infty)$.

Dedicated to Professor Leonard L. Scott on the occasion of his 65th birthday

1. Introduction

The Lie algebra $\mathfrak{gl}_\infty$ of infinite matrices and its extension A_∞ are interesting topics in the theory of infinite dimensional Lie algebras. Certain highest weight representations of these algebras have important applications in finding solutions of a large class of nonlinear equations (see, e.g., [3]) and in the representations theories of Heisenberg algebras, the Virasoro algebra and other Kac-Moody algebras (see, e.g., [17, 18, 27, 28]). The quantum versions of the corresponding universal enveloping algebras have also been studied in [20, 29, 30]. It should be noted that the structure and representations of quantum $\mathfrak{gl}_\infty$ have various connections with the study of Lie superalgebras $\mathfrak{gl}(m|n)$ (see, e.g., [2]) and the study of quantum affine $\mathfrak{gl}_n$ (see [14, §6] and [26, §2]).

In this paper, we will investigate the quantum group $\mathbf{U}(\infty)=\mathbf{U}_v(\mathfrak{gl}_\infty)$ and its representations through a series of its quotient algebras $\mathbf{U}(\infty,r)$ and the infinite quantum Schur algebras $\boldsymbol{\mathcal{S}}(\infty,r)$. The main idea is to extend the approach developed in [1] for quantum $\mathfrak{gl}_n$ to quantum $\mathfrak{gl}_\infty$. Thus, we obtain a realization for $\mathbf{U}(\infty)$ and an explicit description of the algebra homomorphism $\zeta_r:\mathbf{U}(\infty)\to\mathbf{U}(\infty,r)$. This in turn gives rise to a presentation and various useful bases for $\mathbf{U}(\infty,r)$. We will also prove that $\mathbf{U}(\infty,r)$ is a proper subalgebra of $\boldsymbol{\mathcal{S}}(\infty,r)$. This fact shows that the classical Schur–Weyl duality fails in this case. Finally, we investigate the 'polynomial' representation theory and classify all irreducible polynomial representations for $\mathbf{U}(\infty)$. We expect that this work will have further applications to quantum affine $\mathfrak{gl}_n$.

We organize the paper as follows. We start in §1 with a general construction of two completion algebras $\widehat{\mathcal{A}}$ and $\widehat{\mathcal{A}}^\dagger$ for certain algebras $\mathcal{A}$. We recall the definition of quantum $\mathfrak{gl}_\eta$ and q-Schur algebras $\mathcal{S}(\eta,r)$ at any consecutive segment η of $\mathbb{Z}$ in §2. In §3, we use the η-step flag variety to define the algebra $\mathcal{K}(\eta,r)$ and discuss its stabilization property and triangular relations developed in [1] in a context suitable for infinite η. From §4 onwards, we will focus on the infinite case $\eta=\mathbb{Z}$. First, the stabilization property allows us to define an algebra $\boldsymbol{\mathcal{K}}(\infty)$ over $\mathbb{Q}(v)$ whose completion

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$\widehat{\boldsymbol{\mathcal{K}}}(\infty)$ contains a subalgebra $\mathbf{V}(\infty)$ which is isomorphic to $\mathbf{U}(\infty)$. Second, there is a algebra homomorphism $\xi_r : \mathbf{V}(\infty) \to \widehat{\boldsymbol{\mathcal{K}}}(\infty, r)$ with image $\mathbf{V}(\infty, r)$. The elementary structure of $\mathbf{V}(\infty, r)$ is investigated in §5, and a presentation for $\mathbf{V}(\infty, r)$ will be given in §6. In §7, we will establish isomorphisms $\boldsymbol{\mathcal{S}}(\infty, r) \cong \widehat{\boldsymbol{\mathcal{K}}}^{\dagger}(\infty, r)$ and between $\mathbf{V}(\infty, r)$ and the homomorphic image $\mathbf{U}(\infty, r)$ of the natural homomorphism $\zeta_r : \mathbf{U}(\infty) \to \boldsymbol{\mathcal{S}}(\infty, r)$. Thus, we conclude that ζ_r is not surjective for any $r \geqslant 1$. In §8, by identifying $\boldsymbol{\mathcal{K}}(\infty)$ with the modified quantum group $\dot{\mathbf{U}}(\infty)$, we derive an algebra epimorphism $\dot{\zeta}_r : \boldsymbol{\mathcal{K}}(\infty) \to \boldsymbol{\mathcal{K}}(\infty, r)$ and hence, extend the map ζ_r to an epimorphism from $\widehat{\boldsymbol{\mathcal{K}}}^{\dagger}(\infty)$ to $\boldsymbol{\mathcal{S}}(\infty, r)$. In the last three sections, we investigate the representation theory of $\mathbf{U}(\infty)$. The highest weight representations of $\mathbf{U}(\infty)$ is studied in §9, and the polynomial representations of $\mathbf{U}(\infty)$ is in §11 as an application of the representation theory of $\boldsymbol{\mathcal{S}}(\infty, r)$ investigated in §10.

Throughout the paper, we will encounter several (associative) algebras $\mathcal{A}$ over a commutative ring $\mathcal{R}$ without the identity element, but with many orthogonal idempotents e_i, $i \in I$, such that $\mathcal{A} = \oplus_{i,i' \in I} e_i \mathcal{A} e_{i'}$ and, for all $i, i' \in I$, $e_i \mathcal{A} e_{i'}$ are free over $\mathcal{R}$. Clearly, the index set I must be an infinite set, since if I was a finite set, then $\sum_{i \in I} e_i$ would be the identity element of $\mathcal{A}$.

Choose an $\mathcal{R}$-basis $\mathcal{B} = \{a_j\}_{j \in J}$ such that $e_i \mathcal{B} e_{i'} = \{e_i a_j e_{i'}\}_{j \in J} \backslash \{0\}$ is a basis for $e_i \mathcal{A} e_{i'}$ for all $i, i' \in I$, and $\mathcal{B} = \cup_{i,i'} e_i \mathcal{B} e_{i'}$. We further assume that $e_i \in \mathcal{B}$ for all $i \in I$. It is clear that, for any $j \in J$, there exist unique $i_j, i'_j \in I$ such that $e_{i_j} a_j = a_j$ and $a_j e_{i'_j} = a_j$. We will write $ro(j) := i_j$ and $co(j) := i'_j$ for all $j \in J$.

For a formal infinite linear combination $f = \sum_{j \in J} f_j a_j$ with $f_j \in \mathcal{R}$, let, for any $i \in I$,

$$\begin{aligned} J(e_i, f) &:= \{j \in J \mid f_j \neq 0, e_i a_j = a_j\} = \{j \in J \mid f_j \neq 0, ro(j) = i\}, \\ J(f, e_i) &:= \{j \in J \mid f_j \neq 0, a_j e_i = a_j\} = \{j \in J \mid f_j \neq 0, co(j) = i\}. \end{aligned}$$

Lemma 1.1. *Let $\mathfrak{L}$ be the set[1] of formal (possibly infinite) linear combinations of $\mathcal{B}$, and let*

$$\begin{aligned} {}^{\dagger}\widehat{\mathcal{A}} &:= \{f \in \mathfrak{L} \mid \forall i \in I, |J(e_i, f)| < \infty\} \\ \widehat{\mathcal{A}}^{\dagger} &:= \{f \in \mathfrak{L} \mid \forall i \in I, |J(f, e_i)| < \infty\} \\ \widehat{\mathcal{A}} &:= \{f \in \mathfrak{L} \mid \forall i, i' \in I, |J(e_i, f)| < \infty,\ |J(f, e_{i'})| < \infty\} = {}^{\dagger}\widehat{\mathcal{A}} \cap \widehat{\mathcal{A}}^{\dagger}. \end{aligned}$$

We define the product of two elements $\sum_{s \in J} f_s a_s, \sum_{t \in J} g_t a_t$ in $\widehat{\mathcal{A}}^{\dagger}$ (resp., ${}^{\dagger}\widehat{\mathcal{A}}$) to be $\sum_{s,t \in J} f_s g_t a_s a_t$ where $a_s a_t$ is the product in $\mathcal{A}$. Then ${}^{\dagger}\widehat{\mathcal{A}}$ and $\widehat{\mathcal{A}}^{\dagger}$ become associative algebras with identity $1 = \sum_{i \in I} e_i$ and $\widehat{\mathcal{A}}$ is a subalgebra with the same identity.

The algebras ${}^{\dagger}\widehat{\mathcal{A}}$, $\widehat{\mathcal{A}}^{\dagger}$ and $\widehat{\mathcal{A}}$ are called the *completion algebras* of $\mathcal{A}$.

Proof. We need to prove that the multiplication is well-defined. Let $f = \sum_{s \in J} f_s a_s, g = \sum_{t \in J} g_t a_t \in \widehat{\mathcal{A}}^{\dagger}$. If $s, t \in J$ is such that $co(s) \neq ro(t)$ then $a_s a_t = a_s e_{co(s)} e_{ro(t)} a_t = 0$. Hence, we have

$$fg = \sum_{s,t \in J} f_s g_t a_s a_t = \sum_{i \in I} x_i, \quad \text{where} \quad x_i = \sum_{\substack{t \in J \\ co(t) = i}} \sum_{\substack{s \in J \\ co(s) = ro(t)}} f_s g_t a_s a_t.$$

Since $a_s a_t \in \mathcal{A}$ and the set $\{(s,t) \mid f_s g_t \neq 0, s, t \in J, co(s) = ro(t), co(t) = i\}$ is finite, it follows that $x_i \in \mathcal{A}$. Write $x_i = \sum_{\substack{j \in J \\ co(j) = i}} h_{j,i} a_j$ (all $h_{j,i} = 0$ except for finitely many j's), and put $(fg)_j = h_{j,co(j)}$, then $fg = \sum_{i \in I} x_i = \sum_{j \in J} (fg)_j a_j$. Clearly, for each $i \in I$, the set $\{j \in J \mid (fg)_j \neq 0, co(j) = i\}$ is

[1]We may identify the set $\mathfrak{L}$ with the direct product $\prod_{j \in J} \mathcal{R} a_j$.

finite. Hence, $fg \in \widehat{\mathcal{A}}^\dagger$ and so the multiplication is well-defined. It is clear that this multiplication defines an associative algebra structure on $\widehat{\mathcal{A}}^\dagger$.

Similarly, we can show that ${}^\dagger\widehat{\mathcal{A}}$ is also an associative algebra. The last assertion is clear. □

Note that, if $\mathcal{A}$ admits an algebra antiautomorphism f satisfying $f(e_i) = e_i$ for all $i \in I$, then ${}^\dagger\widehat{\mathcal{A}} \cong (\widehat{\mathcal{A}}^\dagger)^{\mathrm{op}}$, the algebras with the same underlying space as $\widehat{\mathcal{A}}^\dagger$ but opposite multiplication.

Some notations and conventions. A *consecutive segment* η of $\mathbb{Z}$ is either a finite interval of the form $[m, n] := \{i \in \mathbb{Z} \mid m \leqslant i \leqslant n\}$ or an infinite interval of the from $(-\infty, m]$, $[n, +\infty)$ and $(-\infty, \infty) = \mathbb{Z}$, where $m, n \in \mathbb{Z}$. Let

$$\eta^{\dashv} = \begin{cases} \eta \backslash \{\eta_{\max}\}, & \text{if there is a maximal element } \eta_{\max} \text{ in } \eta; \\ \eta, & \text{otherwise.} \end{cases}$$

Let $M_\eta(\mathbb{Z})$ (resp. $\mathbb{Z}^\eta$) be the set of all matrices $(a_{i,j})_{i,j\in\eta}$ (resp. all sequences $(a_i)_{i\in\eta}$) over $\mathbb{Z}$ with *finite support*. Thus, $M_\eta(\mathbb{Z})$ (resp., $\mathbb{Z}^\eta$) can be viewed as a free module over $\mathbb{Z}$ and contains the subset $M_\eta(\mathbb{N})$ (resp., $\mathbb{N}^\eta$). We will always abbreviate the sub-/supscript η by n if $\eta = [1, n]$, and by ∞ if $\eta = (-\infty, \infty)$. Thus, $M_\infty(\mathbb{Z}) = M_{(-\infty,\infty)}(\mathbb{Z})$ consists of all $\mathbb{Z} \times \mathbb{Z}$-matrices over $\mathbb{Z}$ of finite support.

For later use in §§9-11, let $X(\eta) = \prod_{i\in\eta} \mathbb{Z}$ (a direct product of $|\eta|$ copies $\mathbb{Z}$). Clearly, $\mathbb{Z}^\infty$ is a subset of $X(\infty)$, and $X(\eta) = \mathbb{Z}^\eta$ if and only if η is finite.

Let $\widetilde{\Xi}(\eta) = \{(a_{ij}) \in M_\eta(\mathbb{Z}) \mid a_{ij} \geqslant 0 \ \forall i \neq j\}$ and let $\Xi(\eta) = M_\eta(\mathbb{N}) := \{(a_{ij}) \in M_\eta(\mathbb{Z}) \mid a_{ij} \geqslant 0\}$. For any $A = (a_{ij}) \in \Xi(\eta)$ (resp. $\mathbf{j} = (j_i) \in \mathbb{N}^\eta$), let $\sigma(A) = \sum_{i,j} a_{ij}$ (resp., $\sigma(\mathbf{j}) = \sum_i j_i$). We also set $\Xi(\eta, r) = \{A \in \Xi(\eta) \mid \sigma(A) = r\}$ for any $r \geqslant 0$.

Let $\Xi^\pm(\eta)$ be the set of all $A \in \Xi(\eta)$ whose diagonal entries are zero. Let $\Xi^+(\eta)$ (resp., $\Xi^-(\eta)$) be the subset of $\Xi(\eta)$ consisting of those matrices (a_{ij}) with $a_{ij} = 0$ for all $i \geqslant j$ (resp., $i \leqslant j$). Let $\Xi^0(\eta)$ (resp., $\widetilde{\Xi}^0(\eta)$) denotes the subset of diagonal matrices in $\Xi(\eta)$ (resp., $\widetilde{\Xi}(\eta)$). For $A \in \widetilde{\Xi}(\eta)$, write $A = A^+ + A^0 + A^- = A^\pm + A^0$ with $A^+ \in \Xi^+(\eta)$, $A^- \in \Xi^-(\eta)$, $A^0 \in \widetilde{\Xi}^0(\eta)$ and $A^\pm = A^+ + A^- \in \Xi^\pm(\eta)$. For $A \in \Xi(\eta)$ let

$$\deg(A) = \sum_{i,j\in\mathbb{Z}} |j - i| a_{ij}. \tag{1.1.1}$$

If $\eta \subseteq \eta'$, there is natural embedding from $M_\eta(\mathbb{Z})$ to $M_{\eta'}(\mathbb{Z})$, sending $A \in M_\eta(\mathbb{Z})$ to $A^{\eta'} = A{\uparrow}^{\eta'} \in M_{\eta'}(\mathbb{Z})$ obtained by adding 0's at the (i, j) positions for all (i, j) with either $i \in \eta' \backslash \eta$ or $j \in \eta' \backslash \eta$. We call $A^{\eta'}$ the *extension of A by zeros*. For notational simplicity, we often identify $A^{\eta'}$ as A and write $A \in M_{\eta'}(\mathbb{Z})$. Thus, it is convenient to regard any $M_\eta(\mathbb{Z})$ as a subset of $M_{\eta'}(\mathbb{Z})$ whenever $\eta \subseteq \eta'$. In particular, every $M_\eta(\mathbb{Z})$ can be regarded as a subset of $M_\infty(\mathbb{Z})$. With this convention, we have $\widetilde{\Xi}(\infty) = \cup_{n\geqslant 0} \widetilde{\Xi}([-n, n])$, $\Xi(\infty) = \cup_{n\geqslant 0} \Xi([-n, n])$ and $\Xi(\infty, r) = \cup_{n\geqslant 0} \Xi([-n, n], r)$. Similarly, there is natural restriction from $M_{\eta'}(\mathbb{Z})$ to $M_\eta(\mathbb{Z})$, sending $B \in M_{\eta'}(\mathbb{Z})$ to $B_\eta = B{\downarrow}_\eta \in M_\eta(\mathbb{Z})$ obtained by deleting all rows and columns labelled by elements in $\eta' \backslash \eta$. We will simply write $B \in M_\eta(\mathbb{Z})$ if $B = B{\downarrow}_\eta{\uparrow}^{\eta'}$.

We will adopt similar conventions for the sets $\mathbb{Z}^\eta$.

2. Quantum $\mathfrak{gl}_\eta$ and q-Schur algebras at η

In order to give a unified definition for both quantum $\mathfrak{gl}_n$ and quantum $\mathfrak{gl}_\infty$, we introduce $\mathfrak{gl}_\eta$ for any consecutive segment η of $\mathbb{Z}$.

Definition 2.1. The quantum $\mathfrak{gl}_\eta$ over $\mathbb{Q}(\upsilon)$ is the $\mathbb{Q}(\upsilon)$-algebra $\mathbf{U}(\eta) := \mathbf{U}_\upsilon(\mathfrak{gl}_\eta)$ presented by generators

$$E_i,\ F_i \quad (i \in \eta^{\dashv}),\ K_j,\ K_j^{-1} \quad (j \in \eta)$$

and relations

(a) $K_iK_j = K_jK_i,\ K_iK_i^{-1} = 1$;
(b) $K_iE_j = \upsilon^{\delta_{i,j}-\delta_{i,j+1}}E_jK_i$;
(c) $K_iF_j = \upsilon^{\delta_{i,j+1}-\delta_{i,j}}F_jK_i$;
(d) $E_iE_j = E_jE_i,\ F_iF_j = F_jF_i$ *when* $|i-j| > 1$;
(e) $E_iF_j - F_jE_i = \delta_{i,j}\frac{\widetilde{K}_i-\widetilde{K}_i^{-1}}{\upsilon-\upsilon^{-1}}$, *where* $\widetilde{K}_i = K_iK_{i+1}^{-1}$;
(f) $E_i^2E_j - (\upsilon+\upsilon^{-1})E_iE_jE_i + E_jE_i^2 = 0$ *when* $|i-j| = 1$;
(g) $F_i^2F_j - (\upsilon+\upsilon^{-1})F_iF_jF_i + F_jF_i^2 = 0$ *when* $|i-j| = 1$.

The algebra $\mathbf{U}(\eta)$ is a Hopf algebra with comultiplication Δ defined on generators by

$$\Delta(E_i) = E_i \otimes \widetilde{K}_i + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + \widetilde{K}_i^{-1} \otimes F_i, \quad \Delta(K_j) = K_j \otimes K_j.$$

For notational simplicity, we set $\mathbf{U}(\eta) = \begin{cases} \mathbf{U}(n), & \text{if } \eta = [1,n]; \\ \mathbf{U}(\infty), & \text{if } \eta = (-\infty,+\infty). \end{cases}$ Clearly, we have natural embedding $\mathbf{U}([-n,n]) \subseteq \mathbf{U}([-n-1,n+1]) \subseteq \mathbf{U}(\infty)$ for all $n \geqslant 0$. Thus, we obtain an algebra isomorphism

$$\mathbf{U}(\infty) = \varinjlim_{n} \mathbf{U}([-n,n]). \tag{2.1.1}$$

Let $\mathbf{U}^+(\eta)$ (resp., $\mathbf{U}^-(\eta)$, $\mathbf{U}^0(\eta)$) be the subalgebra of $\mathbf{U}(\eta)$ generated by the E_i (resp., F_i, $K_j^{\pm 1}$). The subalgebras $\mathbf{U}^+(\eta)$ and $\mathbf{U}^-(\eta)$ are both $\mathbb{N}$-graded in terms of the degrees of monomials in the E_i and F_i. For monomials M in the E_i and M' in the F_i, and an element $h \in \mathbf{U}^0(\eta)$, write $\deg(MhM') = \deg(M) + \deg(M')$. Note that deg does *not* define an algebra grading on $\mathbf{U}(\eta)$. However, if $\Pi(\eta) = \{\alpha_j := \boldsymbol{e}_j - \boldsymbol{e}_{j+1} \mid j \in \eta^{\dashv}\}$ denotes the set of simple roots, where $\boldsymbol{e}_i = (\cdots,0,\underset{i}{1},0\cdots) \in \mathbb{Z}^{\eta}$, then there is an algebra grading over the root lattice $\mathbb{Z}\Pi(\eta)$,

$$\mathbf{U}(\eta) = \bigoplus_{\nu \in \mathbb{Z}\Pi(\eta)} \mathbf{U}(\eta)_\nu \tag{2.1.2}$$

defined by the conditions $\mathbf{U}(\eta)_{\nu'}\mathbf{U}(\eta)_{\nu''} \subseteq \mathbf{U}(\eta)_{\nu'+\nu''}$, $K^{\mathbf{j}} \in \mathbf{U}(\eta)_0$, $E_i \in \mathbf{U}(\eta)_{\alpha_i}$, $F_i \in \mathbf{U}(\eta)_{-\alpha_i}$ for all $\nu', \nu'' \in \mathbb{Z}\Pi(\eta)$, $i \in \eta^{\dashv}$ and $\mathbf{j} \in \mathbb{Z}^\eta$.

Let $\mathcal{Z} = \mathbb{Z}[\upsilon,\upsilon^{-1}]$, and let $[m]^! = [1][2]\cdots[m]$ where $[t] = \frac{\upsilon^t-\upsilon^{-t}}{\upsilon-\upsilon^{-1}}$. The Lusztig $\mathcal{Z}$-form is the $\mathcal{Z}$-subalgebra $U(\eta)$ of $\mathbf{U}(\eta)$ generated by the elements $E_i^{(m)} = \frac{E_i^m}{[m]^!}$, $F_i^{(m)} = \frac{F_i^m}{[m]^!}$, K_j and

$$\begin{bmatrix} K_j;c \\ t \end{bmatrix} = \prod_{s=1}^{t} \frac{K_j\upsilon^{c-s+1} - K_j^{-1}\upsilon^{-c+s-1}}{\upsilon^s - \upsilon^{-s}}, \tag{2.1.3}$$

for all $i \in \eta^{\dashv}$, $j \in \eta$, $m,t \in \mathbb{N}$ and $c \in \mathbb{Z}$. Let $U^+(\eta)$ (resp., $U^-(\eta)$, $U^0(\eta)$) be the subalgebra of $U(\eta)$ generated by the $E_i^{(m)}$ (resp., $F_i^{(m)}$, $K_j^{\pm1}$ and $\begin{bmatrix} K_j;c \\ t \end{bmatrix}$).

We record the following commutator formula in $U(\eta)$; see [21, 4.1(a)].

Lemma 2.2. *For any positive integers k,l, we have*

$$E_i^{(k)}F_i^{(l)} = \sum_{t=0}^{\min(k,l)} F_i^{(l-t)} \begin{bmatrix} \widetilde{K}_i; 2t-k-l \\ t \end{bmatrix} E_i^{(k-t)}.$$

For each $A \in \Xi^{\pm}(\eta)$ and $\mathbf{j} = (j_i)_{i\in\mathbb{Z}} \in \mathbb{Z}^{\eta}$, choose $m, n \in \eta$ such that $m \leqslant n$ and $A \in \Xi^{\pm}([m,n])$ and let

$$E^{(A^+)} = \prod_{m\leqslant i\leqslant h<j\leqslant n}^{(\leqslant_1)} E_h^{(a_{i,j})}, \quad F^{(A^-)} = \prod_{m\leqslant j\leqslant h<i\leqslant n}^{(\leqslant_2)} F_h^{(a_{i,j})} \text{ and } K^{\mathbf{j}} = \prod_{i\in\mathbb{Z}} K_i^{j_i},$$

where $\leqslant_1$ and $\leqslant_2$ indicate the orders in the product which are defined on the set $\{(i,h,j) \mid m \leqslant i \leqslant h < j \leqslant n\}$ by

$$\text{(2.2.1)} \quad \begin{aligned} (i,h,j) \leqslant_1 (i',h',j') &\iff \text{one of the following three conditions is satisfied:} \\ &\quad\ \text{i) } j > j',\ \text{ii) } j=j',\ i>i', \text{ or iii) } j=j',\ i=i',\ h\leqslant h'; \\ (i,h,j) \leqslant_2 (i',h',j') &\iff \text{one of the following three conditions is satisfied:} \\ &\quad\ \text{i) } j < j',\ \text{ii) } j=j',\ i<i', \text{ or iii) } j=j',\ i=i',\ h\geqslant h'. \end{aligned}$$

In other words, $E^{(A^+)} = M_n M_{n-1} \cdots M_{m+1}$ and $F^{(A^-)} = M'_{m+1} M'_{m+2} \cdots M'_n$, where

$$M_j = E_{j-1}^{(a_{j-1,j})} (E_{j-2}^{(a_{j-2,j})} E_{j-1}^{(a_{j-2,j})}) \cdots (E_m^{(a_{m,j})} E_{m+1}^{(a_{m,j})} \cdots E_{j-1}^{(a_{m,j})}),$$

and

$$M'_j = (F_{j-1}^{(a_{j,m})} \cdots F_{m+1}^{(a_{j,m})} F_m^{(a_{j,m})}) \cdots (F_{j-1}^{(a_{j,j-2})} F_{j-2}^{(a_{j,j-2})}) F_{j-1}^{(a_{j,j-1})}.$$

It is clear that $E^{(A^+)}$ and $F^{(A^-)}$ are independent of the selection of m, n. Note that we have, for $A \in \Xi^{\pm}(\eta)$, $\deg(E^{(A^+)}) = \deg(A^+)$ and $\deg(F^{(A^-)}) = \deg(A^-)$.

The following result is due to Lusztig (see, e.g., [22, 2.14], [1, 5.7] and [9]); cf. (2.1.1) if η is infinite.

Proposition 2.3. (1) *The set* $\{E^{(A^+)} K^{\mathbf{j}} F^{(A^-)} \mid A \in \Xi^{\pm}(\eta), \mathbf{j} \in \mathbb{Z}^{\eta}\}$ *forms a* $\mathbb{Q}(\upsilon)$*-basis for* $\mathbf{U}(\eta)$.

(2) *The set*

$$\left\{ E^{(A^+)} \prod_{i\in\eta} K_i^{\delta_i} \begin{bmatrix} K_i;0 \\ t_i \end{bmatrix} F^{(A^-)} \;\middle|\; A \in \Xi^{\pm}(\eta), \mathbf{t} \in \mathbb{N}^{\eta}, (\delta_i)_{i\in\eta} \in \mathbb{N}^{\eta}, \delta_i \in \{0,1\} \text{ for } i \in \eta \right\}$$

forms a $\mathcal{Z}$*-basis for* $U(\eta)$.

We now extend the definition of q-Schur algebras (or quantum Schur algebras). Let Ω_η be a free $\mathcal{Z}$-module with basis $\{\omega_i\}_{i\in\eta}$. Let $\mathbf{\Omega}_\eta = \Omega_\eta \otimes \mathbb{Q}(\upsilon)$. Then $\mathbf{U}(\eta)$ acts on $\mathbf{\Omega}_\eta$ naturally defined by

$$K_a\omega_b = \upsilon^{\delta_{a,b}}\omega_b \,(a,b\in\eta), \quad E_a\omega_b = \delta_{a+1,b}\omega_a, \quad F_a\omega_b = \delta_{a,b}\omega_{a+1} \,(a \in \eta^{\dashv}, b\in\eta).$$

The tensor space $\mathbf{\Omega}_\eta^{\otimes r}$ is a $\mathbf{U}(\eta)$-module via the comultiplication Δ on $\mathbf{U}(\eta)$. Further, restriction gives the $U(\eta)$-module $\Omega_\eta^{\otimes r}$.

Let $\mathcal{H}$ be the Hecke algebra over $\mathcal{Z}$ associated with the symmetric group $\mathfrak{S}_r$, and let $\boldsymbol{\mathcal{H}} = \mathcal{H} \otimes_{\mathcal{Z}} \mathbb{Q}(\upsilon)$. Thus, $\mathcal{H}$ as a $\mathcal{Z}$-algebra has a basis $\{\mathcal{T}_w\}_{w\in\mathfrak{S}_r}$ subject the relations: for all $w \in \mathfrak{S}_r$ and $s \in S := \{(i,i+1) \mid 1 \leqslant i \leqslant r-1\}$

$$\mathcal{T}_s\mathcal{T}_w = \begin{cases} \mathcal{T}_{sw}, & \text{if } \ell(sw) > \ell(w); \\ (\upsilon - \upsilon^{-1})\mathcal{T}_w + \mathcal{T}_s w, & \text{if } \ell(sw) < \ell(w), \end{cases}$$

where ℓ is the length function on $\mathfrak{S}_r$ with respect to S.

The Hecke algebra $\mathcal{H}$ acts on $\Omega_\eta^{\otimes r}$ on the right by "place permutations" via

$$(\omega_{i_1}\cdots\omega_{i_r})\mathcal{T}_{(j,j+1)} = \begin{cases} \omega_{i_1}\cdots\omega_{i_{j+1}}\omega_{i_j}\cdots\omega_{i_r}, & \text{if } i_j < i_{j+1}; \\ \upsilon\omega_{i_1}\cdots\omega_{i_r}, & \text{if } i_j = i_{j+1}; \\ (\upsilon-\upsilon^{-1})\omega_{i_1}\cdots\omega_{i_r} + \omega_{i_1}\cdots\omega_{i_{j+1}}\omega_{i_j}\cdots\omega_{i_r}, & \text{if } i_j > i_{j+1}. \end{cases}$$

The endomorphism algebras

$$\boldsymbol{\mathcal{S}}(\eta,r) := \mathrm{End}_{\boldsymbol{\mathcal{H}}}(\boldsymbol{\Omega}_\eta^{\otimes r}), \qquad \mathcal{S}(\eta,r) := \mathrm{End}_{\mathcal{H}}(\Omega_\eta^{\otimes r})$$

are called *q-Schur algebras at* (η,r). Note that $\boldsymbol{\mathcal{S}}(n,r) := \boldsymbol{\mathcal{S}}(\eta,r)$ for $\eta=[1,n]$ is usually called the *q-Schur algebra* of bidegree (n,r). In general, if $n=|\eta|$ is finite, $\boldsymbol{\mathcal{S}}(\eta,r)\cong\boldsymbol{\mathcal{S}}(n,r)$ and $\boldsymbol{\mathcal{S}}(\eta,r)\cong\mathcal{S}(\eta,r)\otimes_{\mathcal{Z}}\mathbb{Q}(\upsilon)$. If $\eta=\mathbb{Z}$, $\boldsymbol{\mathcal{S}}(\infty,r):=\boldsymbol{\mathcal{S}}(\eta,r)$ is called an *infinite q-Schur algebra*. We shall use similar notations for their integral versions.

Since the $\boldsymbol{\mathcal{H}}$-action commutes with the action of $\mathbf{U}(\eta)$ (see, e.g., [2]), we obtain algebra homomorphisms

$$\zeta_r : \mathbf{U}(\eta) \longrightarrow \boldsymbol{\mathcal{S}}(\eta,r), \qquad \zeta_r|_{U(\eta)} : U(\eta) \longrightarrow \mathcal{S}(\eta,r). \tag{2.3.1}$$

Let $\mathbf{U}(\eta,r)=\mathrm{Im}(\zeta_r)$ and $U(\eta,r)=\mathrm{Im}(\zeta_r|_{U(\eta)})$. It is known that both maps are surjective if η is finite. However, this is not the case when η is infinite; see 7.4 below. Thus, for $\eta=(-\infty,\infty)$ both $\mathbf{U}(\infty,r)$ and $U(\infty,r)$ are proper subalgebras. We will investigate the relationship between $\mathbf{U}(\infty,r)$ and $\boldsymbol{\mathcal{S}}(\infty,r)$ in §7 via the BLM type realizations for $\mathbf{U}(\infty)$ and $\mathbf{U}(\infty,r)$ discussed in the next few sections.

3. The algebra $\mathcal{K}(\eta,r)$ and its stabilization property

In this section, we review the geometric construction of the q-Schur algebra and extend it to the infinite case.

Let V be a vector space of dimension r over a field k. An *η-step flag* is a collection $\mathfrak{f}=(V_i)_{i\in\eta}$ of subspaces of V such that $V_i\subseteq V_{i+1}$ for all $i\in\eta^{-|}$ and $\bigcup_{i\in\eta}V_i=V$ and $V_i=0$ for $i\ll 0$ if η has no minimal element. If η has a minimal element $\eta_{\min}$, let $V_{\eta_{\min}-1}=0$.

Let $\mathcal{F}$ be the set of η-step flags. The group $G:=GL(V)$ acts naturally on $\mathcal{F}$, and hence, diagonally on $\mathcal{F}\times\mathcal{F}$. For $(\mathfrak{f},\mathfrak{f}')\in\mathcal{F}\times\mathcal{F}$, where $\mathfrak{f}=(V_i)_{i\in\eta}$ and $\mathfrak{f}'=(V_i')_{i\in\eta}$, we let

$$a_{i,j}=\dim(V_{i-1}+(V_i\cap V_j'))-\dim(V_{i-1}+(V_i\cap V_{j-1}')).$$

Then the map $(\mathfrak{f},\mathfrak{f}')\mapsto(a_{i,j})$ induces a bijection from the set of G-orbits on $\mathcal{F}\times\mathcal{F}$ to the set $\Xi(\eta,r)$. Let $\mathcal{O}_A\subseteq\mathcal{F}\times\mathcal{F}$ be the G-orbit corresponding to the matrix $A\in\Xi(\eta,r)$.

When k is a finite field of q-elements, every orbit $\mathcal{O}_A$ is a finite set. Thus, for any $A,B,C\in\Xi(\eta,r)$ and any fixed $(\mathfrak{f}_1,\mathfrak{f}_2)\in\mathcal{O}_C$, the number

$$g_{A,B,C;q} := \{\mathfrak{f}\in\mathcal{F}\mid(\mathfrak{f}_1,\mathfrak{f})\in\mathcal{O}_A,(\mathfrak{f},\mathfrak{f}_2)\in\mathcal{O}_B\}$$

is independent of the selection of $(\mathfrak{f}_1,\mathfrak{f}_2)$. We have clearly

$$g_{A,B,C;q}=0 \ \text{ unless } co(A)=ro(B), \tag{3.0.2}$$

where, for a matrix $C=(c_{i,j})\in M_\eta(\mathbb{Z})$

$$ro(C)=\big(\Sigma_j c_{i,j}\big)_{i\in\eta}\in\mathbb{N}^\eta \ \text{ and } \ co(C)=\big(\Sigma_i c_{i,j}\big)_{j\in\eta}\in\mathbb{N}^\eta \tag{3.0.3}$$

are the sequences of row and column sums of C.

It is well-known that there exists a polynomial $g_{A,B,C}\in\mathbb{Z}[\upsilon^2]$ such that $g_{A,B,C}|_{\upsilon^2=q}=g_{A,B,C;q}$ for any q.

Let $\mathcal{K}(\eta,r)$ be the free $\mathcal{Z}$-module with basis $\{e_A\mid A\in\Xi(\infty,r)\}$. By [1, 1.2] there is a associative $\mathcal{Z}$-algebra structure on $\mathcal{K}(\eta,r)$ with multiplication $e_A\cdot e_B=\sum_{C\in\Xi(\eta,r)}g_{A,B,C}e_C$. Note that, if η is finite, then $\mathcal{K}(\eta,r)$ is isomorphic to the q-Schur algebra $\mathcal{S}(|\eta|,r)$ (see, e.g., [9]). However, when η is infinite, the algebra $\mathcal{K}(\eta,r)$ has no identity element.

For any $n\geqslant 1$, there is a natural injective map from $\mathcal{K}([-n,n],r)$ to $\mathcal{K}([-n-1,n+1],r)$ by sending $[A]$ to $[A]$ for any $A\in\Xi([-n,n],r)\subseteq\Xi([-n-1,n+1],r)$. Note that this map preserves

multiplication, but it does not send 1 to 1. Thus, we obtain a direct system $\{\mathcal{K}([-n,n],r)\}_{n\geqslant 1}$. It is clear

$$\mathcal{K}(\infty,r):=\mathcal{K}(\mathbb{Z},r)=\varinjlim_n \mathcal{K}([-n,n],r).$$

Let $^-:\mathcal{Z}\to\mathcal{Z}$ denote the ring homomorphism sending υ to υ^{-1}. For integers N,t with $t\geqslant 1$ let

$$\left[\!\!\left[N \atop t \right]\!\!\right]=\prod_{1\leqslant i\leqslant t}\frac{\upsilon^{2(N-i+1)}-1}{\upsilon^{2i}-1}\in\mathcal{Z}.$$

Let

$$\Lambda(\eta,r)=\{\lambda\in\mathbb{N}^\eta\mid\sigma(\lambda)=r\},\tag{3.0.4}$$

and let $\leqslant$ be the partial order on $\Lambda(\eta,r)$ defined by setting, for $\lambda,\mu\in\Lambda(\eta,r)$,

$$\lambda\leqslant\mu \text{ if and only if } \lambda_i\leqslant\mu_i \text{ for all } i\in\eta.\tag{3.0.5}$$

We further consider the basis $\{[A]\}_{A\in\Xi(\eta,r)}$ for $\mathcal{K}(\eta,r)$, where

$$[A]:=\upsilon^{-d_A}e_{A}\quad\text{with}\quad d_A=\sum_{i\geqslant k,j<l}a_{ij}a_{kl}.\tag{3.0.6}$$

The following explicit multiplication formulas is given in [1, 3.4] for $n\times n$-matrices. For convenience, we now restate it to include infinite matrices with finite support.

Proposition 3.1. *Let η be a consecutive segment of $\mathbb{Z}$ and let $A\in\Xi(\eta,r)$. Suppose $h\in\mathbb{Z}$ and $b\geqslant 0$.*

(a) If $B\in\Xi(\eta,r)$ satisfies that $B-bE_{h,h+1}$ is a diagonal matrix and $co(B)=ro(A)$, then

$$[B]\cdot[A]=\sum_{\substack{\nu\in\Lambda(\eta,b)\\ \nu\leqslant row_{h+1}A}}\upsilon^{\beta(\nu,A)}\prod_{i\in\mathbb{Z}}\overline{\left[\!\!\left[a_{h,i}+\nu_i \atop \nu_i \right]\!\!\right]}\Big[A+\sum_{i\in\mathbb{Z}}\nu_i(E_{h,i}-E_{h+1,i})\Big]$$

where $\beta(\nu,A)=\sum_{j\geqslant i}a_{h,j}\nu_i-\sum_{j>i}a_{h+1,j}\nu_i+\sum_{i<i'}\nu_i\nu_{i'}$.

(b) If $C\in\Xi(\eta,r)$ satisfies that $C-bE_{h+1,h}$ is a diagonal matrix and $co(B)=ro(A)$, then

$$[B]\cdot[A]=\sum_{\substack{\nu\in\Lambda(\eta,b)\\ \nu\leqslant row_{h}A}}\upsilon^{\gamma(\nu,A)}\prod_{i\in\mathbb{Z}}\overline{\left[\!\!\left[a_{h+1,i}+\nu_i \atop \nu_i \right]\!\!\right]}\Big[A-\sum_{i\in\mathbb{Z}}\nu_i(E_{h,i}-E_{h+1,i})\Big]$$

where $\gamma(\nu,A)=\sum_{j\leqslant i}a_{h+1,j}\nu_i-\sum_{j<i}a_{h,j}\nu_i+\sum_{i<i'}\nu_i\nu_{i'}$.

Let υ' be another indeterminate (independent of υ) and for $A\in\widetilde{\Xi}(\eta)$ and $\nu\in\Lambda(\eta,r)$, let

$$P_{\nu,A}(\upsilon,\upsilon')=\upsilon^{\beta(\nu,A)}\prod_{i\in\mathbb{Z},i\neq h}\overline{\left[\!\!\left[a_{h,i}+\nu_i \atop \nu_i \right]\!\!\right]}\cdot\prod_{j=1}^{\nu_h}\frac{\upsilon^{-2(a_{h,h}+\nu_h-j+1)}\upsilon'^2-1}{\upsilon^{-2j}-1}$$

and

$$Q_{\nu,A}(\upsilon,\upsilon')=\upsilon^{\gamma(\nu,A)}\prod_{i\in\mathbb{Z},i\neq h+1}\overline{\left[\!\!\left[a_{h+1,i}+\nu_i \atop \nu_i \right]\!\!\right]}\cdot\prod_{j=1}^{\nu_{h+1}}\frac{\upsilon^{-2(a_{h+1,h+1}+\nu_{h+1}-j+1)}\upsilon'^2-1}{\upsilon^{-2j}-1}$$

These are polynomials in $\mathbb{Q}(\upsilon)[\upsilon']$.

For a finite segment η of $\mathbb{Z}$, let $I_\eta\in\Xi(\eta)$ be the identity matrix of size $|\eta|$. Consider $A\in\widetilde{\Xi}([m_0,n_0])$. For any $a,m,n\in\mathbb{Z}$ with $m\leqslant m_0$ and $n_0\leqslant n$, let ${}_a(A^{[m,n]})=A^{[m,n]}+aI_{[m,n]}\in\widetilde{\Xi}([m,n])$. Clearly, when a is large enough, ${}_a(A^{[m,n]})\in\Xi([m,n])$. For each $A\in\widetilde{\Xi}(\eta)$ and any $\nu\in\Lambda(\eta,m)$ we define $\beta(\nu,A)$ and $\gamma(\nu,A)$ by the same formulas as given in 3.1. One checks easily that, if

$A \in \widetilde{\Xi}([m_0, n_0])$, then $\beta(\nu, {}_a(A^{[m,n]})) = \beta(\nu, A)$ and $\gamma(\nu, {}_a(A^{[m,n]})) = \gamma(\nu, A)$ for all a and any $m \leqslant m_0$, $n \geqslant n_0$.

Corollary 3.2. *Let A, B be the matrices in $\widetilde{\Xi}([m_0, n_0])$ such that $co(B) = ro(A)$.*

(1) If $B - bE_{h,h+1}$ is diagonal, then, for all $m \leqslant m_0$, $n \geqslant n_0$ and $a \gg 0$, we have

$$[{}_a(B^{[m,n]})] \cdot [{}_a(A^{[m,n]})] = \sum_{\substack{\nu \in \Lambda(\eta,b) \\ \nu_j \leqslant a_{h+1,j}, \forall j \neq h+1}} P_{\nu,A}(\upsilon, \upsilon^{-a}) \big[{}_a\big(A^{[m,n]} + \sum_i \nu_i (E^{[m,n]}_{h,i} - E^{[m,n]}_{h+1,i})\big)\big]$$

(2)If $B - bE_{h+1,h}$ is diagonal, then, for all $m \leqslant m_0$, $n \geqslant n_0$ and $a \gg 0$, we have

$$[{}_a(B^{[m,n]})] \cdot [{}_a(A^{[m,n]})] = \sum_{\substack{\nu \in \Lambda(\eta,b) \\ \nu_j \leqslant a_{h,j}, \forall j \neq h}} Q_{\nu,A}(\upsilon, \upsilon^{-a}) \big[{}_a\big(A^{[m,n]} - \sum_i \nu_i (E^{[m,n]}_{h,i} - E^{[m,n]}_{h+1,i})\big)\big]$$

For every $A = (a_{s,t}) \in \widetilde{\Xi}(\eta)$ and $i \neq j$, let

$$\sigma_{i,j}(A) = \begin{cases} \sum_{s \leqslant i; t \geqslant j} a_{s,t}, & \text{if } i < j; \\ \sum_{s \geqslant i; t \leqslant j} a_{s,t}, & \text{if } i > j. \end{cases} \tag{3.2.1}$$

Define $B \preccurlyeq A$ if and only if $\sigma_{i,j}(B) \leqslant \sigma_{i,j}(A)$ for all $i, j \in \eta$, $i \neq j$. Put $B \prec A$ if $B \preccurlyeq A$ and $\sigma_{i,j}(B) < \sigma_{i,j}(A)$ for some $i \neq j$.

There is another order $\leqslant$ on $\Xi(n, r)$, called the *Bruhat order*, which is defined by setting $B \leqslant A$ if $\mathcal{O}_B \subseteq \overline{\mathcal{O}_A}$, the closure of $\mathcal{O}_A$. By [1, 3.6(b)], we have $A \leqslant B$ implies $A \preccurlyeq B$, and $B \preccurlyeq A$ and $A \preccurlyeq B$ imply $A = B$.

The following is a modified version of the important triangular relations given in [1, 3.9] with the Bruhat order replaced by the order $\preccurlyeq$, where the notation $[C]+$ *low terms* means $[C]$ plus a $\mathcal{Z}$-linear combination of elements $[B]$ with $B \in \Xi(\eta, r)$, $B \prec C$. This result is required in establishing the generalized stabilization property 3.4.

Proposition 3.3. *For any given $m_0 \leqslant n_0$ in $\mathbb{Z}$ and $A \in \widetilde{\Xi}([m_0, n_0])$, choose an integer $b_0 \geqslant 0$ such that $A + b_0 I_{m_0,n_0} \in \Xi([m_0, n_0])$. For any $b, m, n \in \mathbb{Z}$ with $b \geqslant b_0$, $m \leqslant m_0$ and $n \geqslant n_0$, the following identity:*

$$\prod^{(\leqslant_1)}_{m_0 \leqslant i \leqslant h < j \leqslant n_0} [D_{i,h,j} + a_{ij} E_{h,h+1} + bI_{[m,n]}] \cdot \prod^{(\leqslant_2)}_{m_0 \leqslant j \leqslant h < i \leqslant n_0} [D_{i,h,j} + a_{ij} E_{h+1,h} + bI_{[m,n]}]$$
$$= [A + bI_{[m,n]}] + \textit{lower terms relative to } \preccurlyeq,$$

holds in $\mathcal{K}([m, n], r)$, where $r = \sigma(A) + b(n - m + 1)$. Here the ordering of the products is the same as in (2.2.1), and the matrices $D_{i,h,j}$ are certain diagonal matrices, determined by A and independent of m, n and b.

Proof. Regarding A as a matrix in $\widetilde{\Xi}([m, n])$, then $B := A + bI_{[m,n]} \in \Xi([m, n], r)$. Now, applying [1, 3.9] to the matrix B yields the formula

$$\prod^{(\leqslant_1)}_{m_0 \leqslant i \leqslant h < j \leqslant n_0} [\widetilde{D}_{i,h,j} + a_{ij} E_{h,h+1}] \cdot \prod^{(\leqslant_2)}_{m_0 \leqslant j \leqslant h < i \leqslant n_0} [\widetilde{D}_{i,h,j} + a_{ij} E_{h+1,h}]$$
$$= [B] + \textit{lower terms relative to } \leqslant,$$

where $\widetilde{D}_{i,h,j}$ are diagonal matrices determined recursively as follows: $\widetilde{D}_{n-1,n-1,n}$ is determined by the condition $ro(B) = ro(\widetilde{D}_{n-1,n-1,n} + a_{n-1,n} E_{n-1,n})$ and, if (i', h', j') is an immediate predecessor of (i, h, j) in the product, then $\widetilde{D}_{i,h,j}$ is determined by the condition $co(\widetilde{D}_{i',h',j} + a_{i',j'} E') = ro(D_{i,h,j} +$

$a_{i,j}E$), where E', E represent matrices of the form $E_{a,a+1}$ or $E_{a+1,a}$. (For example, $\widetilde{D}_{n-2,n-2,n}$ is determined by the condition $co(\widetilde{D}_{n-1,n-1,n} + a_{n-1,n}E_{n-1,n}) = ro(D_{n-2,n-2,n} + a_{n-2,n}E_{n-2,n-1})$ and so on.) Clearly, each diagonal entry of $\widetilde{D}_{i,h,j}$ is a sum of the corresponding diagonal entry of B and some of the off diagonal entries of B. So, if we put $D_{i,h,j} = \widetilde{D}_{i,h,j} - bI_{[m,n]}$, then $D_{i,h,j}$ is a sum of the corresponding diagonal entry of A and some of the off diagonal entries of A. Hence, $D_{i,h,j}$ is independent of m, n and b. Since $A \leqslant B$ implies $A \preccurlyeq B$, substitution gives the required formula. □

Let $\mathcal{Z}_1$ be the subring of $\mathbb{Q}(\upsilon)[\upsilon']$ generated by υ^j ($j \in \mathbb{Z}$) and

$$\prod_{1\leqslant i\leqslant t} \frac{\upsilon^{-2(a-i)}\upsilon'^2 - 1}{\upsilon^{-2i} - 1} \quad \text{for } a \in \mathbb{Z},\, t \geqslant 1.$$

We have the following generalized version of the *stabilization property* [1, 4.2].

Theorem 3.4. *Let $m_0, n_0 \in \mathbb{Z}$, $m_0 \leqslant n_0$, and let A, B be matrices in $\widetilde{\Xi}([m_0, n_0])$ such that $co(A) = ro(B)$. Then there exist $X_i \in \widetilde{\Xi}([m_0, n_0])$, elements $P_i(\upsilon, \upsilon') \in \mathcal{Z}_1$ ($1 \leqslant i \leqslant s$) and an integer $a_0 \geqslant 0$ such that*

$$[{}_a(A^{[m,n]})][{}_a(B^{[m,n]})] = \sum_{i=1}^{s} P_i(\upsilon, \upsilon^{-a})[{}_a(X_i^{[m,n]})]$$

for all $a \geqslant a_0$, $m \leqslant m_0$ and $n \geqslant n_0$.

Proof. If A is a matrix such that $A - bE_{h,h+1}$ or $A - bE_{h+1,h}$ is diagonal for some b, h, then the result follows from 3.2. Using 3.3, one can prove the result by induction on $\|A\|$, where[2]

$$\|A\| = \sum_{m\leqslant i<j\leqslant n} \binom{j-i+1}{2}(a_{i,j} + a_{j,i}), \tag{3.4.1}$$

as proceeded in the proof of [1, 4.2]. □

We see from the theorem that the polynomials $P_i(\upsilon, \upsilon')$ depends only on A, B and X_i, and not on any embedding $\widetilde{\Xi}([m_0, n_0]) \subseteq \widetilde{\Xi}(\eta)$ for any segment $\eta \supseteq [m_0, n_0]$. Hence, we have established the following.

Corollary 3.5. *Let η be a fixed consecutive segment of $\mathbb{Z}$. For any $A, B, C \in \widetilde{\Xi}(\eta)$ with $co(A) = ro(B)$, there exist $f_{A,B,C}(\upsilon, \upsilon') \in \mathcal{Z}_1$ such that, if $A, B, C \in \widetilde{\Xi}([m_0, n_0])$ for some $m_0 \leqslant n_0$ in $\mathbb{Z}$ with $[m_0, n_0] \subseteq \eta$, then*

$$[{}_a(A^{[m,n]})] \cdot [{}_a(B^{[m,n]})] = \sum_{C\in\widetilde{\Xi}([m_0,n_0])} f_{A,B,C}(\upsilon, \upsilon^{-a})[{}_a(C^{[m,n]})]$$

for all $m \leqslant m_0$, $n \geqslant n_0$ with $[m, n] \subseteq \eta$, and $a \gg 0$.

4. The BLM realization of $\mathbf{U}(\infty)$

We now use the polynomials $f_{A,B,C}(\upsilon, \upsilon')$ given in 3.4 to define a new associative algebra.

[2] $\|A\|$ is denoted by $\Psi(A)$ in [1].

Consider the free $\mathcal{Z}_1$-module with basis $\{A \mid A \in \widetilde{\Xi}(\eta)\}$. Define a multiplication on this module by linearly extending the products on basis elements (cf. (3.0.2)):

$$A \cdot B := \begin{cases} \sum_{C \in \widetilde{\Xi}(\eta)} f_{A,B,C}(v, v')C, & \text{if } co(A) = ro(B) \\ 0 & \text{otherwise} \end{cases}$$

Then we get an associative algebra over $\mathcal{Z}_1$ which has no identity element.

By specializing $v' = 1$, we get an associative $\mathcal{Z}$-algebra $\mathcal{K}(\eta)$ with basis $[A] := A \otimes 1$ ($A \in \widetilde{\Xi}(\eta)$) in which the product $[A] \cdot [B]$ is given by $\sum_{C \in \widetilde{\Xi}(\eta)} f_{A,B,C}(v, 1)[C]$, if $co(A) = ro(B)$, and it is zero, otherwise. In particular, we have in $\mathcal{K}(\eta)$, for any $D \in \Xi^0(\eta)$ (and so $co(D) = ro(D)$),

$$[D][A] = \begin{cases} [A] & \text{if } co(D) = ro(A) \\ 0 & \text{otherwise;} \end{cases} \quad \text{and} \quad [A][D] = \begin{cases} [A] & \text{if } ro(D) = co(A) \\ 0 & \text{otherwise.} \end{cases} \tag{4.0.1}$$

There is a natural algebra embedding from $\mathcal{K}([-n, n])$ into $\mathcal{K}([-n-1, n+1])$ induced by the embedding $\widetilde{\Xi}([-n, n]) \subseteq \widetilde{\Xi}([-n-1, n+1])$. Thus, we obtain a direct system $\{\mathcal{K}([-n, n])\}_{n \geqslant 1}$. It is clear we have

$$\mathcal{K}(\infty) = \varinjlim_n \mathcal{K}([-n, n]).$$

Let $\boldsymbol{\mathcal{K}}(\eta) = \mathcal{K}(\eta) \otimes_{\mathcal{Z}} \mathbb{Q}(v)$ and $\boldsymbol{\mathcal{K}}(\eta, r) = \mathcal{K}(\eta, r) \otimes_{\mathcal{Z}} \mathbb{Q}(v)$. The set $\{[\mathrm{diag}(\lambda)] \mid \lambda \in \mathbb{Z}^\eta\}$ (resp., $\{[\mathrm{diag}(\lambda)] \mid \lambda \in \Lambda(\eta, r)\}$) is a set of orthogonal idempotents of $\boldsymbol{\mathcal{K}}(\eta)$ (resp., $\boldsymbol{\mathcal{K}}(\eta, r)$). By 1.1 we may construct the completion algebra $\widehat{\boldsymbol{\mathcal{K}}}(\eta)$ (resp., $\widehat{\boldsymbol{\mathcal{K}}}(\eta, r)$) of $\boldsymbol{\mathcal{K}}(\eta)$ (resp., $\boldsymbol{\mathcal{K}}(\eta, r)$). The element $\sum_{\lambda \in \mathbb{Z}^\eta} [\mathrm{diag}(\lambda)]$ (resp., $\sum_{\lambda \in \Lambda(\eta, r)} [\mathrm{diag}(\lambda)]$) is the identity element of $\widehat{\boldsymbol{\mathcal{K}}}(\eta)$ (resp., $\widehat{\boldsymbol{\mathcal{K}}}(\eta, r)$). Note that if η is a finite set, then we have $\boldsymbol{\mathcal{K}}(\eta, r) = \widehat{\boldsymbol{\mathcal{K}}}(\eta, r)$.

Given $r > 0$, $A \in \Xi^{\pm}(\eta)$ and $\mathbf{j} \in \mathbb{Z}^\eta$, we define

$$\begin{aligned} A(\mathbf{j}, r) = A(\mathbf{j}, r)_\eta &= \sum_{\substack{\lambda \in \mathbb{N}^\eta \\ \sigma(A) + \sigma(\lambda) = r}} v^{\lambda \cdot \mathbf{j}} [A + \mathrm{diag}(\lambda)] \in \widehat{\boldsymbol{\mathcal{K}}}(\eta, r), \\ A(\mathbf{j}) = A(\mathbf{j})_\eta &= \sum_{\lambda \in \mathbb{Z}^\eta} v^{\lambda \cdot \mathbf{j}} [A + \mathrm{diag}(\lambda)] \in \widehat{\boldsymbol{\mathcal{K}}}(\eta). \end{aligned} \tag{4.0.2}$$

where $\lambda \cdot \mathbf{j} = \sum_{i \in \mathbb{Z}} \lambda_i j_i < \infty$. Note that these summations depend on η.

For $i \in \eta$, $j \in \eta^{\dashv}$, let as in §2

$$\boldsymbol{e}_i = (\cdots, 0, \underset{i}{1}, 0 \cdots) \in \mathbb{Z}^\eta \text{ and } \alpha_j = \boldsymbol{e}_j - \boldsymbol{e}_{j+1} \in \mathbb{Z}^\eta. \tag{4.0.3}$$

Lemma 4.1. *Let $\lambda, \mu \in \mathbb{Z}^\eta$. The following formulas hold in $\widehat{\boldsymbol{\mathcal{K}}}(\eta)$.*

(1) *For $i \in \eta^{\dashv}$ we have $E_{i,i+1}(\mathbf{0})[\mathrm{diag}(\lambda)] = [\mathrm{diag}(\lambda + \alpha_i)]E_{i,i+1}(\mathbf{0})$;*
(2) *For $i \in \eta^{\dashv}$ we have $E_{i+1,i}(\mathbf{0})[\mathrm{diag}(\lambda)] = [\mathrm{diag}(\lambda - \alpha_i)]E_{i+1,i}(\mathbf{0})$.*

Let $\mathbf{V}(\eta)$ (resp. $\mathbf{V}(\eta, r)$) be the subspace of $\widehat{\boldsymbol{\mathcal{K}}}(\eta)$ (resp. $\widehat{\boldsymbol{\mathcal{K}}}(\eta, r)$) spanned by the elements $A(\mathbf{j})$ (resp., $A(\mathbf{j}, r)$) for $A \in \Xi^{\pm}(\eta)$ and $\mathbf{j} \in \mathbb{Z}^\eta$. It is clear that the elements $A(\mathbf{j})$ ($A \in \Xi^{\pm}(\eta)$, $\mathbf{j} \in \mathbb{Z}^\eta$) form a $\mathbb{Q}(v)$-basis of $\mathbf{V}(\eta)$. There are similar bases for $\mathbf{V}(\eta, r)$; see [10] and Corollary 6.7 below.

When $\eta = [m, n]$ is *finite*, it is known from [1, 5.7] that $\mathbf{V}([m, n])$ is a subalgebra of $\widehat{\boldsymbol{\mathcal{K}}}([m, n])$ and there is an isomorphism

$$\mathbf{V}([m, n]) \cong \mathbf{U}([m, n]) \quad \text{via} \quad E_h \mapsto E_{h,h+1}(\mathbf{0}), \quad K^{\mathbf{j}} \mapsto 0(\mathbf{j}), \quad F_h \mapsto E_{h+1,h}(\mathbf{0}). \tag{4.1.1}$$

Moreover, $\boldsymbol{\mathcal{K}}([m, n], r) = \mathbf{V}([m, n], r) = \widehat{\boldsymbol{\mathcal{K}}}([m, n], r)$ is a q-Schur algebra in this case.

When η is *infinite*, we will see below that the isomorphism $\mathbf{U}(\eta) \cong \mathbf{V}(\eta)$ continues to hold, but the equalities $\boldsymbol{\mathcal{K}}([m,n],r) = \mathbf{V}([m,n],r) = \widehat{\boldsymbol{\mathcal{K}}}([m,n],r)$ are replaced by a chain relation $\boldsymbol{\mathcal{K}}(\eta,r) \subseteq \mathbf{V}(\eta,r) \subseteq \widehat{\boldsymbol{\mathcal{K}}}(\eta,r)$; see 4.3(2) and 5.5 below. Moreover, $\mathbf{V}(\eta,r)$ is isomorphic to $\mathbf{U}(\eta,r)$.

For simplicity, we will only consider the case when $\eta = \mathbb{Z}$ in the sequel.

Observe that the embedding from $\mathcal{K}([-n,n])$ into $\mathcal{K}([-n-1,n+1])$ induces an algebra embedding from $\mathbf{V}([-n,n])$ into $\mathbf{V}([-n-1,n+1])$. Hence, we obtain a direct system $\{\mathbf{V}([-n,n])\}_{n\geqslant 1}$. It is natural to expect that there is an algebra isomorphism (see the proof of 4.3(1) below)

$$\mathbf{V}(\infty) \cong \varinjlim_n \mathbf{V}([-n,n]). \tag{4.1.2}$$

This isomorphism together with (2.1.1) and (4.1.1) establish an isomorphism $\mathbf{U}(\infty) \cong \mathbf{V}(\infty)$.

The key to the isomorphism (4.1.2) is to show that the embedding from $\mathbf{V}([-n,n])$ to $\widehat{\boldsymbol{\mathcal{K}}}(\infty)$, sending $A(\mathbf{j})_{[-n,n]}$ to $A(\mathbf{j})_\infty$ for any $A \in \Xi^\pm([-n,n])$ and $\mathbf{j} \in \mathbb{Z}^{[-n,n]}$, is an algebra homomorphism. The following multiplication formulas will guarantee this.

Proposition 4.2. *For $i,h \in \mathbb{Z}$, $\mathbf{j},\mathbf{j}' \in \mathbb{Z}^\infty$ and $A \in \Xi^\pm(\infty)$, if we put $\beta_h = -\boldsymbol{e}_h - \boldsymbol{e}_{h+1}$, $f(i) = f(i,A) = \sum_{j\geqslant i} a_{h,j} - \sum_{j>i} a_{h+1,j}$ and $f'(i) = f'(i,A) = \sum_{j<i} a_{h,j} - \sum_{j\leqslant i} a_{h+1,j}$, then the following multiplication identities hold in $\widehat{\boldsymbol{\mathcal{K}}}(\infty)$:*

$$\begin{aligned} 0(\mathbf{j})A(\mathbf{j}') &= v^{\sum_{i,k} j_i a_{i,k}} A(\mathbf{j}+\mathbf{j}') \\ A(\mathbf{j}')0(\mathbf{j}) &= v^{\sum_{i,k} j_i a_{k,i}} A(\mathbf{j}+\mathbf{j}') \end{aligned} \tag{4.2.1}$$

where 0 *stands for the zero matrix.*

$$\begin{aligned} E_{h,h+1}(\mathbf{0})A(\mathbf{j}) = &\sum_{i<h;\,a_{h+1,i}\geqslant 1} v^{f(i)} \overline{\left[\!\!\left[\begin{matrix} a_{h,i}+1 \\ 1 \end{matrix} \right]\!\!\right]} (A + E_{h,i} - E_{h+1,i})(\mathbf{j}+\alpha_h) \\ &+ \sum_{i>h+1;\,a_{h+1,i}\geqslant 1} v^{f(i)} \overline{\left[\!\!\left[\begin{matrix} a_{h,i}+1 \\ 1 \end{matrix} \right]\!\!\right]} (A + E_{h,i} - E_{h+1,i})(\mathbf{j}) \\ &+ \alpha v^{f(h)-j_h-1} \frac{(A - E_{h+1,h})(\mathbf{j}+\alpha_h) - (A - E_{h+1,h})(\mathbf{j}+\beta_h)}{1-v^{-2}} \\ &+ v^{f(h+1)+j_{h+1}} \overline{\left[\!\!\left[\begin{matrix} a_{h,h+1}+1 \\ 1 \end{matrix} \right]\!\!\right]} (A + E_{h,h+1})(\mathbf{j}) \end{aligned} \tag{4.2.2}$$

where α is 1 *if $a_{h+1,h} \geqslant 1$ and is* 0 *otherwise.*

$$\begin{aligned} E_{h+1,h}(\mathbf{0})A(\mathbf{j}) = &\sum_{i<h;\,a_{h,i}\geqslant 1} v^{f'(i)} \overline{\left[\!\!\left[\begin{matrix} a_{h+1,i}+1 \\ 1 \end{matrix} \right]\!\!\right]} (A - E_{h,i} + E_{h+1,i})(\mathbf{j}) \\ &+ \sum_{i>h+1;\,a_{h,i}\geqslant 1} v^{f'(i)} \overline{\left[\!\!\left[\begin{matrix} a_{h+1,i}+1 \\ 1 \end{matrix} \right]\!\!\right]} (A - E_{h,i} + E_{h+1,i})(\mathbf{j}-\alpha_h) \\ &+ \alpha' v^{f'(h+1)-j_{h+1}-1} \frac{(A - E_{h,h+1})(\mathbf{j}-\alpha_h) - (A - E_{h,h+1})(\mathbf{j}+\beta_h)}{1-v^{-2}} \\ &+ v^{f'(h)+j_h} \overline{\left[\!\!\left[\begin{matrix} a_{h+1,h}+1 \\ 1 \end{matrix} \right]\!\!\right]} (A + E_{h+1,h})(\mathbf{j}) \end{aligned} \tag{4.2.3}$$

where α' is 1 *if $a_{h,h+1} \geqslant 1$ and is* 0 *otherwise.*

The same formulas hold in the algebra $\widehat{\boldsymbol{\mathcal{K}}}(\infty,r)$ with $A(\mathbf{j})$ replaced by $A(\mathbf{j},r)$.

Proof. By 3.1 one can derive formulas similar to [1, 4.6(a),(b)]. The remaining proof is the same as the proof of [1, 5.3]. □

Note that, though we may regard a finite matrix A as a matrix in $\Xi(\infty)$, we cannot regard $A(\mathbf{j})_{[-n,n]}$ in $\mathbf{V}([-n,n])$ as the element $A(\mathbf{j})_\infty$ in $\mathbf{V}(\infty)$. Thus, the formulas above appear in the same form as, but cannot be derived from, those given in [1, 5.3].

Theorem 4.3. (1) $\mathbf{V}(\infty)$ *is a subalgebra of* $\widehat{\boldsymbol{\mathcal{K}}}(\infty)$. *Moreover, the algebra* $\mathbf{U}(\infty)$ *is isomorphic to* $\mathbf{V}(\infty)$ *by sending* E_h *to* $E_{h,h+1}(\mathbf{0})$, F_h *to* $E_{h+1,h}(\mathbf{0})$, *and* $K^{\mathbf{j}}$ *to* $0(\mathbf{j})$ *where* $h \in \mathbb{Z}$ *and* $\mathbf{j} \in \mathbb{Z}^{\infty}$.

(2) *There is an algebra homomorphism*

$$\xi_r : \mathbf{V}(\infty) \longrightarrow \widehat{\boldsymbol{\mathcal{K}}}(\infty, r)$$

by sending $A(\mathbf{j})$ *to* $A(\mathbf{j}, r)$ *for any* $A \in \Xi^{\pm}(\infty)$ *and* $\mathbf{j} \in \mathbb{Z}^{\infty}$. *In particular, we have* $\mathbf{V}(\infty, r) = \xi_r(\mathbf{V}(\infty))$ *is a subalgebra of* $\widehat{\boldsymbol{\mathcal{K}}}(\infty, r)$.

Proof. Let f be the injective linear map from $\mathbf{V}([-n,n])$ to $\mathbf{V}(\infty)$ by sending $A(\mathbf{j})$ to $A(\mathbf{j})_\infty$ for any $A \in \Xi^{\pm}([-n,n])$ and $\mathbf{j} \in \mathbb{Z}^{[-n,n]}$. Let $X := \{E_{h,h+1}(\mathbf{0}), E_{h+1,h}, 0(\mathbf{j}) \mid -n \leqslant h < n, \mathbf{j} \in \mathbb{Z}^{[-n,n]}\}$. By [1, 5.3] and 4.2 we know $f(xA(\mathbf{j})) = f(x)f(A(\mathbf{j}))$ for any $x \in X$ and $A \in \Xi^{\pm}([-n,n])$ and $\mathbf{j} \in \mathbb{Z}^{[-n,n]}$. Since the algebra $\mathbf{V}([-n,n])$ is generated by X; see (4.1.1), it follows that f is an algebra homomorphism. Thus, $\mathbf{V}(\infty)$ is the direct limit of $\mathbf{V}([-n,n])$ and, hence, is a subalgebra of $\widehat{\boldsymbol{\mathcal{K}}}(\infty)$, proving (1). The statement (2) can be proved similarly. □

Remark 4.4. We remark that the theorem can be proved directly. Using 3.3, we may generalize [1, 4.6(c)] to derive the formula in $\widehat{\boldsymbol{\mathcal{K}}}(\infty)$ (cf. [1, 5.5(c)])

$$\prod_{i\leqslant h<j}(a_{i,j}E_{h,h+1})(\mathbf{0})\prod_{i\leqslant h<j}(a_{j,i}E_{h+1,h})(\mathbf{0}) = A(\mathbf{0}) + f \quad (\forall A \in \widetilde{\Xi}^{\pm}(\infty)) \tag{4.4.1}$$

where f is a linear combination of $B(\mathbf{j})$ with $B \prec A$. Thus, the proof of [1, 5.5] can be carried over.

With the above result, we shall identify $\mathbf{U}(\infty)$ with $\mathbf{V}(\infty)$ in the sequel. Thus, $E_h = E_{h,h+1}(\mathbf{0})$ and $F_h = E_{h+1,h}(\mathbf{0})$ for all $h \in \mathbb{Z}$, and by 4.2

$$E_h^{(m)} = (mE_{h,h+1})(\mathbf{0}), \qquad F_h^{(m)} = (mE_{h+1,h})(\mathbf{0}) \qquad \text{for all } m \geqslant 1. \tag{4.4.2}$$

Hence, the left hand side of (4.4.1) equals $E^{(A^+)}F^{(A^-)}$.

Corollary 4.5. *For any* $A \in \widetilde{\Xi}(\infty)$ *and* $\lambda \in \mathbb{Z}^\infty$, *we have*

$$E^{(A^+)}[\mathrm{diag}(\lambda)] = [\mathrm{diag}(\lambda - co(A^+) + ro(A^+))]E^{(A^+)}$$
$$F^{(A^-)}[\mathrm{diag}(\lambda)] = [\mathrm{diag}(\lambda + co(A^-) - ro(A^-))]F^{(A^-)}$$

In particular, if $\lambda = \boldsymbol{\sigma}(A) := (\sigma_i(A))_{i\in\mathbb{Z}}$, *where* $\sigma_i(A) = a_{i,i} + \sum_{j<i}(a_{i,j} + a_{j,i})$, *then*

$$E^{(A^+)}[\mathrm{diag}(\lambda)] = [\mathrm{diag}(ro(A))]E^{(A^+)}, \qquad [\mathrm{diag}(\lambda)]F^{(A^-)} = F^{(A^-)}[\mathrm{diag}(co(A))].$$

We end this section with a discussion on a new basis for $\mathcal{K}(\infty)$, involving elements in the integral form $U(\infty)$.

Proposition 4.6. *The set* $\mathcal{L} = \{E^{(A^+)}[\mathrm{diag}(\boldsymbol{\sigma}(A))]F^{(A^-)} \mid A \in \widetilde{\Xi}(\infty)\}$ *forms a* $\mathcal{Z}$*-basis for* $\mathcal{K}(\infty)$.

Proof. By 4.5, we have

$$M^{(A)} := E^{(A^+)}[\mathrm{diag}(\boldsymbol{\sigma}(A))]F^{(A^-)} = [\mathrm{diag}(ro(A))]E^{(A^+)}F^{(A^-)}[\mathrm{diag}(co(A))].$$

Thus, (4.4.1) implies $M^{(A)} = [A] + f$, where $f = \sum_{B\in\widetilde{\Xi}(\infty),B\prec A} f_{B,A}[B]$ with $f_{B,A} \in \mathcal{Z}$. Hence, the set is linearly independent. To see the set spans, we proceed by induction on $\|A\|$; see (3.4.1). If $\|A\| = 0$ then A is a diagonal matrix, and so $[A] = [\mathrm{diag}(\boldsymbol{\sigma}(A))] \in \mathcal{L}$. Now we assume that $\|A\| > 0$ and that $[A']$ is a linear combination of $M^{(B)}$ ($B \in \widetilde{\Xi}(\infty)$) for $A' \in \widetilde{\Xi}(\infty)$ with $\|A'\| < \|A\|$. Since $B \prec A$ implies that $\|B\| < \|A\|$ (see [1, p.668]), by induction, f is a linear combination of $M^{(C)}$, $C \in \widetilde{\Xi}(\infty)$. Thus, each $[A]$ ($A \in \widetilde{\Xi}(\infty)$) is a linear combination of $M^{(C)}$, and hence, $\mathcal{L}$ spans $\mathcal{K}(\infty)$. □

5. Elementary structure of $\mathbf{V}(\infty, r)$

We first investigate the elementary structure of $\mathbf{V}(\infty, r)$ as a preparation for obtaining its presentation in the next section. Though this algebra is infinite dimensional, it shares many properties with the q-Schur algebra. But, some of the proofs below are different from those given in [12].

For $i \in \mathbb{Z}$, let

$$\mathtt{k}_i = 0(\boldsymbol{e}_i, r), \ \ \mathtt{e}_i = E_{i,i+1}(\mathbf{0}, r) \quad \text{and} \quad \mathtt{f}_i = E_{i+1,i}(\mathbf{0}, r),$$

and let, for any $\mathbf{t} \in \mathbb{N}^\infty$,

$$\mathtt{k}(\mathbf{t}) = \prod_{i\in\mathbb{Z}} [\mathtt{k}_i; t_i]^! \quad \text{and} \quad \mathtt{k}_{\mathbf{t}} = \prod_{i\in\mathbb{Z}} \begin{bmatrix} \mathtt{k}_i; 0 \\ t_i \end{bmatrix},$$

where $[x; t]^! = (x - 1)(x - \upsilon) \cdots (x - \upsilon^{t-1})$, $[x; 0]^! = 1$ and $\left[\begin{smallmatrix} \mathtt{k}_i;0 \\ t_i \end{smallmatrix}\right]$ is defined as in (2.1.3). Since $\mathbf{t}$ has finite support, the products are well-defined. By Theorem 4.3, $\mathtt{e}_i, \mathtt{k}_i, \mathtt{f}_i$ ($i \in \mathbb{Z}$) generate $\mathbf{V}(\infty, r)$.

Lemma 5.1. (1) *We have* $\mathtt{k}(\mathbf{t}) = 0$ *for all* $\mathbf{t} \in \mathbb{N}^\infty$ *with* $\sigma(\mathbf{t}) > r$.

(2) *For any* $\mathbf{t} \in \mathbb{N}^\infty$, *we have* $\mathtt{k}_{\mathbf{t}} = \begin{cases} 0, & \text{if } \sigma(\mathbf{t}) > r; \\ [\mathrm{diag}(\mathbf{t})], & \text{if } \sigma(\mathbf{t}) = r. \end{cases}$ *In particular, if* $\lambda \in \Lambda(\infty, r)$, *then* $[\mathrm{diag}(\lambda)] = \mathtt{k}_\lambda \in \mathbf{V}(\infty, r)$.

(3) *For any* $\lambda \in \Lambda(\infty, r)$, *we have* $\mathtt{k}_i\mathtt{k}_\lambda = \upsilon^{\lambda_i}\mathtt{k}_\lambda$ *and* $\left[\begin{smallmatrix} \mathtt{k}_i;c \\ t \end{smallmatrix}\right] \mathtt{k}_\lambda = \left[\begin{smallmatrix} \lambda_i+c \\ t \end{smallmatrix}\right]$.

Proof. Since $\sum_{\lambda\in\Lambda(\infty,r)}[\mathrm{diag}(\lambda)]$ is the identity element of $\mathbf{V}(\infty, r)$, we have, for all $i \in \mathbb{Z}$,

$$[\mathtt{k}_i; t_i]^! = \sum_{\lambda\in\Lambda(\infty,r)} [\mathtt{k}_i; t_i]^![\mathrm{diag}(\lambda)] = \sum_{\lambda\in\Lambda(\infty,r)} [\upsilon^{\lambda_i}; t_i]^![\mathrm{diag}(\lambda)].$$

Hence we have

$$\prod_{i\in\mathbb{Z}} [\mathtt{k}_i; t_i]^! = \prod_{i\in\mathbb{Z}} \sum_{\lambda\in\Lambda(\infty,r)} [\upsilon^{\lambda_i}; t_i]^![\mathrm{diag}(\lambda)] = \sum_{\lambda\in\Lambda(\infty,r)} \left(\prod_{i\in\mathbb{Z}} [\upsilon^{\lambda_i}; t_i]^! \right) [\mathrm{diag}(\lambda)].$$

Note that $\prod_{i\in\mathbb{Z}}[\upsilon^{\lambda_i}; t_i]^! \neq 0$ if and only if $\lambda_i \geqslant t_i$ for all $i \in \mathbb{Z}$. If $\sigma(\mathbf{t}) > r$, then, for any $\lambda \in \Lambda(\infty, r)$, there exist $i \in \mathbb{Z}$ such that $\lambda_i < t_i$. Hence $\prod_{i\in\mathbb{Z}}[\mathtt{k}_i; t_i]^! = 0$, proving (1).

Similarly, we have, for $i \in \mathbb{Z}$

$$\begin{bmatrix} \mathtt{k}_i; 0 \\ t_i \end{bmatrix} = \sum_{\lambda\in\Lambda(\infty,r)} \begin{bmatrix} \mathtt{k}_i; 0 \\ t_i \end{bmatrix} [\mathrm{diag}(\lambda)] = \sum_{\lambda\in\Lambda(\infty,r)} \begin{bmatrix} \lambda_i \\ t_i \end{bmatrix} [\mathrm{diag}(\lambda)].$$

Hence we have

$$\mathtt{k}_{\mathbf{t}} = \prod_{i\in\mathbb{Z}} \sum_{\lambda\in\Lambda(\infty,r)} \begin{bmatrix} \lambda_i \\ t_i \end{bmatrix} [\mathrm{diag}(\lambda)] = \sum_{\substack{\lambda\in\Lambda(\infty,r) \\ \lambda_i\geqslant t_i \, for \, all \, i}} \left(\prod_{i\in\mathbb{Z}} \begin{bmatrix} \lambda_i \\ t_i \end{bmatrix} \right) [\mathrm{diag}(\lambda)].$$

If $\sigma(\mathbf{t}) > r$, then for any $\lambda \in \Lambda(\infty, r)$, there exist $i \in \mathbb{Z}$ such that $\lambda_i < t_i$. Hence $\mathtt{k_t} = 0$. Now we assume $\sigma(\mathbf{t}) = r$. Then, for $\lambda \in \Lambda(\infty, r)$ with $\lambda_i \geqslant t_i$ for $i \in \mathbb{Z}$, we have $\lambda_i = t_i$ for $i \in \mathbb{Z}$. Hence $\mathtt{k_t} = [\mathrm{diag}(\mathbf{t})]$, proving (2). (3) is clear from the proof above. □

Let $\mathbf{V}^0(\infty, r)$ be the subalgebra of $\mathbf{V}(\infty, r)$ generated by the $\mathtt{k}_i$, $i \in \mathbb{Z}$.

Corollary 5.2. (1) *The set* $\{\mathtt{k}_\lambda \mid \lambda \in \Lambda(\infty, r)\}$ *is a set of orthogonal idempotents for* $\mathbf{V}(\infty, r)$ *and forms a basis for* $\mathbf{V}^0(\infty, r)$.

(2) *For* $\mathbf{j} \in \mathbb{Z}^\infty$, *write* $\mathtt{k}^{\mathbf{j}} = \prod_{i\in\mathbb{Z}} \mathtt{k}_i^{j_i}$. *Then the set* $\{\mathtt{k}^{\mathbf{j}} \mid \mathbf{j} \in \mathbb{N}^\infty, \sigma(\mathbf{j}) \leqslant r\}$ *forms a basis for* $\mathbf{V}^0(\infty, r)$.

Proof. The first statement is clear. We now prove (2). By 5.1, we know that $[\mathtt{k}_i; r+1]^! = 0$. Hence, $\mathbf{V}^0(\infty, r)$ is spanned by the elements $\prod_{i\in\mathbb{Z}} \mathtt{k}_i^{j_i}$ ($\mathbf{j} \in \mathbb{N}^\infty$, $\sigma(\mathbf{j}) \leqslant r$). Since $\mathtt{k}_i = \sum_{\lambda\in\Lambda(\infty,r)} \upsilon^{\lambda_i}[\mathrm{diag}(\lambda)]$, we have, for any $n \geqslant 1$ and $j_{-n}, \cdots, j_{n-1} \in \mathbb{N}$,

$$\prod_{-n\leqslant i\leqslant n-1} \mathtt{k}_i^{j_i} = \sum_{\lambda\in\Lambda([-n,n],r)} \upsilon^{\sum_{i=-n}^{n-1}\lambda_i j_i}[\mathrm{diag}(\lambda)] + \sum_{\substack{\lambda\notin\Lambda([-n,n],r)\\ \lambda\in\Lambda(\infty,r)}} \upsilon^{\sum_{i=-n}^{n-1}\lambda_i j_i}[\mathrm{diag}(\lambda)] \tag{5.2.1}$$

Since $\det\left(\upsilon^{\sum_{i=-n}^{n-1}\lambda_i j_i}\right)_{\lambda\in\Lambda([-n,n],r),\mathbf{j}\in\Lambda'_{n,r,0}} \neq 0$ by [10, (6.6.2)], it follows that, for any fixed n, the elements $\prod_{i=-n}^{n-1} \mathtt{k}_i^{j_i}$ ($\mathbf{j} \in \mathbb{N}^{[-n,n-1]}, \sigma(\mathbf{j}) \leqslant r$) are linearly independent. Hence, the elements $\prod_{i\in\mathbb{Z}} \mathtt{k}_i^{j_i}$ ($\mathbf{j} \in \mathbb{N}^\infty$, $\sigma(\mathbf{j}) \leqslant r$) are linearly independent. The result follows. □

We record the handy result [10, (6.6.2)]) for later use. Recall the notation $\boldsymbol{\sigma}(A)$ defined in 4.5.

Lemma 5.3. *For* $A \in \Xi^\pm([-n, n])$, *let*

$$\Lambda_{n,r,A} = \{\lambda \in \Lambda([-n,n],r) \mid \lambda \geqslant \boldsymbol{\sigma}(A)\} \quad \text{and} \quad \Lambda'_{n,r,A} = \{\mathbf{j} \in \mathbb{N}^{[-n,n-1]} \mid \sigma(\mathbf{j}) + \sigma(A) \leqslant r\}.$$

Then $|\Lambda_{n,r,A}| = |\Lambda'_{n,r,A}|$ *and* $\det\left(\upsilon^{\sum_{i=-n}^{n-1}\lambda_i j_i}\right)_{\lambda\in\Lambda_{n,r,A},\mathbf{j}\in\Lambda'_{n,r,A}} \neq 0$.

The following results give some commutator relations; cf. [12, 4.8,4.9], 4.1 and 4.5.

Proposition 5.4. *For any* $\lambda \in \Lambda(\infty, r)$ *and* $i \in \mathbb{Z}$, *we have*

$$\mathtt{e}_i\mathtt{k}_\lambda = \begin{cases} \mathtt{k}_{\lambda+\alpha_i}\mathtt{e}_i, & \text{if } \lambda + \alpha_i \in \Lambda(\infty, r); \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad \mathtt{f}_i\mathtt{k}_\lambda = \begin{cases} \mathtt{k}_{\lambda-\alpha_i}\mathtt{f}_i, & \text{if } \lambda - \alpha_i \in \Lambda(\infty, r); \\ 0, & \text{otherwise.} \end{cases}$$

More generally, if $A \in \Xi^\pm(\infty)$ *and* $\lambda \in \Lambda(\infty, r)$, *then we have*

$$\mathtt{e}^{(A^+)}\mathtt{k}_\lambda = \begin{cases} \mathtt{k}_{\lambda-co(A^+)+ro(A^+)}\mathtt{e}^{(A^+)}, & \text{if } \lambda \geqslant \boldsymbol{\sigma}(A^+); \\ 0, & \text{otherwise,} \end{cases}$$

$$\mathtt{f}^{(A^+)}\mathtt{k}_\lambda = \begin{cases} \mathtt{k}_{\lambda+co(A^-)-ro(A^-)}\mathtt{f}^{(A^-)}, & \text{if } \lambda \geqslant \boldsymbol{\sigma}(A^-); \\ 0, & \text{otherwise.} \end{cases}$$

Proof. By 5.1, $\mathtt{e}_i\mathtt{k}_\lambda = E_{i,i+1}(\mathbf{0}, r)[\mathrm{diag}(\lambda)] = \sum_{\mu\in\Lambda(\infty,r-1)}[E_{i,i+1} + \mathrm{diag}(\mu)][\mathrm{diag}(\lambda)]$. Now the first formula follows from (4.0.1). The proof of the second is similar. The last assertion can be proved similarly to the proof of [12, 4.9 and 4.10]. □

For $A \in \Xi(\infty)$ with $\nu = \boldsymbol{\sigma}(A)$, let

$$\mathtt{m}^{(A)} = \mathtt{e}^{(A^+)}\mathtt{k}_\nu\mathtt{f}^{(A^-)}.$$

Theorem 5.5. *For any $A \in \Xi(\infty, r)$, we have the following equation in $\mathbf{V}(\infty, r)$.*

$$\mathtt{m}^{(A)} = [A] + \sum_{B\in\Xi(\infty,r),B\prec A} f_{B,A}[B] \quad \text{(a finite sum)} \tag{5.5.1}$$

where $f_{B,A} \in \mathcal{Z}$. Hence, the elements $\mathtt{m}^{(A)}$, $A \in \Xi(\infty, r)$, form a $\mathcal{Z}$-basis of $\mathcal{K}(\infty, r)$, and $\boldsymbol{\mathcal{K}}(\infty, r) \subseteq \mathbf{V}(\infty, r)$.

Proof. By 5.4 and 5.1(2), we have

$$\begin{aligned}\mathtt{m}^{(A)} &= \mathtt{k}_{\nu+ro(A^+)-co(A^+)}\mathtt{e}^{(A^+)}\mathtt{f}^{(A^-)}\mathtt{k}_{\nu+ro(A^-)-co(A^-)}\\ &= \mathtt{k}_{ro(A)}\mathtt{e}^{(A^+)}\mathtt{f}^{(A^-)}\mathtt{k}_{co(A)} = [\mathrm{diag}(ro(A))]\mathtt{e}^{A^+}\mathtt{f}^{A^-}[\mathrm{diag}(co(A))].\end{aligned}$$

Applying ξ_r given in 4.3 to (4.4.1) yields

$$\mathtt{e}^{(A^+)}\mathtt{f}^{(A^-)} = A^\pm(\mathbf{0}, r) + g, \tag{5.5.2}$$

where g is a linear combination of $B(\mathbf{j}, r)$ with $B \prec A$. Thus, we obtain

$$\mathtt{m}^{(A)} = [A] + f,$$

where f is a linear combination of $[B]$ with $B \prec A$. Hence, the set $\{\mathtt{m}^{(A)}\}_{A\in\Xi(\infty,r)}$ is linearly independent. The proof for span is entirely similar to the way proceeded in the proof of 4.6. □

In general, let, for $A \in \Xi^\pm(\infty)$ and $\lambda \in \Lambda(\infty, r)$, let $\mathtt{m}^{(A,\lambda)} = \mathtt{e}^{(A^+)}\mathtt{k}_\lambda\mathtt{f}^{(A^-)}$.

Corollary 5.6. *Suppose $\mathtt{m}^{(A,\lambda)} \neq 0$ for some $A \in \Xi^\pm(\infty)$ and $\lambda \in \Lambda(\infty, r)$. If there exists $D \in \Xi^0(\infty)$ such that $co(A + D) = \lambda + co(A^-) - ro(A^-)$, then $\mathtt{m}^{(A,\lambda)} = \mathtt{m}^{(A+D)}$. Otherwise,*

$$\mathtt{m}^{(A,\lambda)} = \sum_{B\in\Xi(\infty,r),B\prec A} f'_{B,A}\mathtt{m}^{(B)} \quad \text{(a finite sum).} \tag{5.6.1}$$

where $f'_{B,A} \in \mathbb{Q}(v)$.

Proof. Using (5.5.1), the result can be proved similarly to the proof of [12, 5.6]. □

We end this section by showing that Borel subalgebras of (finite dimensional) q-Schur algebras are natural subalgebras of $\mathbf{V}(\infty, r)$, a fact which plays a crucial role in the determination of a presentation for $\mathbf{V}(\infty, r)$.

Let $\mathbf{V}^{\geqslant 0}([-n, n], r)$ (resp. $\mathbf{V}^{\leqslant 0}([-n, n], r)$) be the subalgebras of $\mathbf{V}([-n, n], r)$ generated by $\mathbf{e}'_i = E_{i,i+1}(\mathbf{0}, r)_{[-n,n]}$ (resp. $\mathbf{f}'_i = E_{i+1,i}(\mathbf{0}, r)_{[-n,n]}$) and $\mathbf{k}'_j = 0(\boldsymbol{e}_j, r)_{[-n,n]}$ with $i \in [-n, n-1]$ and $j \in [-n, n]$. Then, by [12, 8.1], the algebra $\mathbf{V}^{\geqslant 0}([-n, n], r)$ is generated by the elements $\mathbf{e}'_i, \mathbf{k}'_i$, $(-n \leqslant i, j \leqslant n - 1)$, subject to the following relations:

(a) $\mathbf{k}'_i\mathbf{k}'_j = \mathbf{k}'_j\mathbf{k}'_i$;
(b) $[\mathbf{k}'_{-n}; t_{-n}]^![\mathbf{k}'_{-n+1}; t_{-n+1}]^! \cdots [\mathbf{k}'_{n-1}; t_{n-1}]^! = 0$, $\forall t_i \in \mathbb{N}, t_{-n} + \cdots + t_{n-1} = r + 1$;
(c) $\mathbf{e}'_i\mathbf{e}'_j = \mathbf{e}'_j\mathbf{e}'_i \quad (|i - j| > 1)$;
(d) $(\mathbf{e}'_i)^2\mathbf{e}'_j - (v + v^{-1})\mathbf{e}'_i\mathbf{e}'_j\mathbf{e}'_i + \mathbf{e}'_j(\mathbf{e}'_i)^2 = 0$ when $|i - j| = 1$;
(e) $\mathbf{k}'_i\mathbf{e}'_j = v^{\epsilon(i,j)}\mathbf{e}'_j\mathbf{k}'_i$, where $\epsilon(i, i) = 1$, $\epsilon(i + 1, i) = -1$ and $\epsilon(i, j) = 0$, otherwise.

For $\lambda \in \Lambda([-n, n], r)$, let

$$\mathbf{k}'_\lambda = \prod_{i=-n}^{n} \begin{bmatrix} \mathbf{k}'_i; 0 \\ \lambda_i \end{bmatrix} \in \mathbf{V}^{\geqslant 0}([-n, n], r) \cap \mathbf{V}^{\leqslant 0}([-n, n], r)$$

where $\mathbf{k}'_n = v^r\mathbf{k}'^{-1}_{-n} \cdots \mathbf{k}'^{-1}_{n-1}$.

Proposition 5.7. *For $n \geqslant 1$ there are algebra monomorphisms*

$$\varphi_n^{\geqslant 0} : \mathbf{V}^{\geqslant 0}([-n,n],r) \to \mathbf{V}(\infty,r) \quad \textit{satisfying} \quad \mathbf{e}_i' \mapsto \mathtt{e}_i,\ \mathbf{k}_i' \mapsto \mathtt{k}_i \quad (-n \leqslant i \leqslant n-1)$$
$$\varphi_n^{\leqslant 0} : \mathbf{V}^{\leqslant 0}([-n,n],r) \to \mathbf{V}(\infty,r) \quad \textit{satisfying} \quad \mathbf{f}_i' \mapsto \mathtt{f}_i,\ \mathbf{k}_i' \mapsto \mathtt{k}_i \quad (-n \leqslant i \leqslant n-1).$$

Moreover, for any $\lambda \in \Lambda([-n,n],r)$, we have $\varphi_n^{\geqslant 0}(\mathbf{k}_\lambda') = \varphi_n^{\leqslant 0}(\mathbf{k}_\lambda') = \mathtt{h}_{\lambda,n}$, where

$$\mathtt{h}_{\lambda,n} = \sum_{\substack{\mu \in \Lambda(\infty,r) \\ \mu_{-n}=\lambda_{-n},\cdots,\mu_{n-1}=\lambda_{n-1}}} [\mathrm{diag}(\mu)]. \tag{5.7.1}$$

Proof. The presentation above gives an algebra homomorphism $\varphi_n^{\geqslant 0} : \mathbf{V}^{\geqslant 0}([-n,n],r) \to \mathbf{V}(\infty,r)$ mapping $\mathbf{e}_i'$ to $\mathtt{e}_i$ and $\mathbf{k}_i'$ to $\mathtt{k}_i$ for $-n \leqslant i \leqslant n-1$. By 5.5 and [12, 8.2] we have $\varphi_n^{\geqslant 0}$ is injective. The proof for $\varphi_n^{\leqslant 0}$ is similar. It remains to prove the last assertion. We have

$$\begin{aligned}\varphi_n^{\geqslant 0}(\mathbf{k}_\lambda') &= \prod_{i=-n}^{n-1} \begin{bmatrix} \mathtt{k}_i; 0 \\ t_i \end{bmatrix} \cdot \begin{bmatrix} \upsilon^r \mathtt{k}_{-n}^{-1}\mathtt{k}_{-n+1}^{-1}\cdots\mathtt{k}_{n-1}^{-1}; 0 \\ \lambda_n \end{bmatrix} \\ &= \sum_{\mu\in\Lambda(\infty,r)} \left(\prod_{i=-n}^{n-1} \begin{bmatrix} \mu_i \\ \lambda_i \end{bmatrix} \begin{bmatrix} r - \sum_{i=-n}^{n-1}\mu_i \\ \lambda_n \end{bmatrix} \right) [\mathrm{diag}(\mu)].\end{aligned}$$

Since $\lambda \in \Lambda([-n,n],r)$ we have $\sum_{i=-n}^{n-1}(\mu_i - \lambda_i) + (r - \sum_{i=-n}^{n-1}\mu_i - \lambda_n) = r - \sum_{i=-n}^{n}\lambda_i = 0$. Hence for $\mu \in \Lambda(\infty,r)$ we have

$$\begin{aligned}\prod_{i=-n}^{n-1} \begin{bmatrix} \mu_i \\ \lambda_i \end{bmatrix} \begin{bmatrix} r - \sum_{i=-n}^{n-1}\mu_i \\ \lambda_n \end{bmatrix} \neq 0 &\iff \mu_i \geqslant \lambda_i \text{ for } -n \leqslant i \leqslant n-1 \text{ and } r - \sum_{i=-n}^{n-1}\mu_i \geqslant \lambda_n \\ &\iff \mu_i = \lambda_i \text{ for } -n \leqslant i \leqslant n-1.\end{aligned}$$

Hence, $\prod_{i=-n}^{n-1}\begin{bmatrix} \mu_i \\ \lambda_i \end{bmatrix}\begin{bmatrix} r-\sum_{i=-n}^{n-1}\mu_i \\ \lambda_n \end{bmatrix} = 1$, and (5.7.1) follows. $\square$

Remark 5.8. The algebra monomorphisms $\varphi_n^{\geqslant 0}$ and $\varphi_n^{\leqslant 0}$ can't be glued as an algebra homomorphism from $\mathbf{V}([-n,n],r)$ to $\mathbf{V}(\infty,r)$ since the last relation in [12, 4.1] does not hold in $\mathbf{V}(\infty,r)$.

6. Presenting $\mathbf{V}(\infty,r)$

We are now ready to describe a presentation for $\mathbf{V}(\infty,r)$.

Definition 6.1. Let $\widetilde{\mathbf{V}}(\infty,r)$ be the associative algebra over $\mathbb{Q}(\upsilon)$ generated by the elements

$$\mathbf{e}_i, \mathbf{f}_i, \mathbf{k}_i \quad (i \in \mathbb{Z}),$$

subject to the relations:

(a) $\mathbf{k}_i\mathbf{k}_j = \mathbf{k}_j\mathbf{k}_i$;
(b) $\prod_{i\in\mathbb{Z}}[\mathbf{k}_i; t_i]^! = 0$ for all $\mathbf{t} \in \mathbb{N}^\infty$ with $\sigma(\mathbf{t}) = r+1$;
(c) $\mathbf{e}_i\mathbf{e}_j = \mathbf{e}_j\mathbf{e}_i, \mathbf{f}_i\mathbf{f}_j = \mathbf{f}_j\mathbf{f}_i \quad (|i-j|>1)$;
(d) $\mathbf{e}_i^2\mathbf{e}_j - (\upsilon+\upsilon^{-1})\mathbf{e}_i\mathbf{e}_j\mathbf{e}_i + \mathbf{e}_j\mathbf{e}_i^2 = 0 \quad (|i-j|=1)$;
(e) $\mathbf{f}_i^2\mathbf{f}_j - (\upsilon+\upsilon^{-1})\mathbf{f}_i\mathbf{f}_j\mathbf{f}_i + \mathbf{f}_j\mathbf{f}_i^2 = 0 \quad (|i-j|=1)$;
(f) $\mathbf{k}_i\mathbf{e}_j = \upsilon^{\delta_{i,j}-\delta_{i,j+1}}\mathbf{e}_j\mathbf{k}_i, \quad \mathbf{k}_i\mathbf{f}_j = \upsilon^{-\delta_{i,j}+\delta_{i,j+1}}\mathbf{f}_j\mathbf{k}_i$;
(g) $\mathbf{e}_i\mathbf{f}_j - \mathbf{f}_j\mathbf{e}_i = \delta_{i,j}\frac{\widetilde{\mathbf{k}}_i - \widetilde{\mathbf{k}}_i^{-1}}{\upsilon-\upsilon^{-1}}$, where $\widetilde{\mathbf{k}}_i = \mathbf{k}_i\mathbf{k}_{i+1}^{-1}$, $i \in \mathbb{Z}$.

Comparing 6.1 with 2.1, we see that there is an algebra epimorphism $\mathbf{U}(\infty) \twoheadrightarrow \widetilde{\mathbf{V}}(\infty, r)$ satisfying $E_i \mapsto \mathbf{e}_i$, $F_i \mapsto \mathbf{f}_i$ and $K_j \mapsto \mathbf{k}_j$. In particular, for $A \in \Xi^\pm(\infty)$, let $\mathbf{e}^{(A^+)}$, $\mathbf{f}^{(A^-)}$, etc., be the images of $E^{(A^+)}$, $F^{(A^-)}$, etc., under this homomorphism.

On the other hand, by 4.3(2) and 5.1(1), there is an algebra epimorphism

$$\pi : \widetilde{\mathbf{V}}(\infty, r) \longrightarrow \mathbf{V}(\infty, r), \tag{6.1.1}$$

defined by $\mathbf{e}_i \mapsto \mathtt{e}_i$, $\quad \mathbf{f}_i \mapsto \mathtt{f}_i$, $\quad \mathbf{k}_i \mapsto \mathtt{k}_i$. We shall prove that π is an isomorphism by displaying a spanning set for $\widetilde{\mathbf{V}}(\infty, r)$ whose image under π is a linearly independent set in $\mathbf{V}(\infty, r)$.

We first have the following counterpart of 5.7 by comparing the above defining relations with the relations for Borel subalgebras.

Lemma 6.2. *For any given $n \geqslant 1$, there are algebra monomorphisms sending 1 to 1*

$$\psi_n^{\geqslant 0} : \mathbf{V}^{\geqslant 0}([-n, n], r) \to \widetilde{\mathbf{V}}(\infty, r) \quad \textit{satisfying} \quad \mathbf{e}'_i \mapsto \mathbf{e}_i,\ \mathbf{k}'_i \mapsto \mathbf{k}_i \quad (-n \leqslant i \leqslant n-1)$$
$$\psi_n^{\leqslant 0} : \mathbf{V}^{\leqslant 0}([-n, n], r) \to \widetilde{\mathbf{V}}(\infty, r) \quad \textit{satisfying} \quad \mathbf{f}'_i \mapsto \mathbf{f}_i,\ \mathbf{k}'_i \mapsto \mathbf{k}_i \quad (-n \leqslant i \leqslant n-1).$$

Proof. The injectivity follows from 5.7 since $\varphi_n^{\geqslant 0}$ is the composition of $\psi_n^{\geqslant 0}$ and π. □

This result shows that the subalgebra $\mathbf{B}_n^+$ (resp. $\mathbf{B}_n^-$) of $\widetilde{\mathbf{V}}(\infty, r)$ generated by $\mathbf{e}_i, \mathbf{k}_i$ (resp. $\mathbf{f}_i, \mathbf{k}_i$), $-n \leqslant i \leqslant n-1$, is a (finite dimensional) Borel subalgebra investigated in [12, §8]. Let $\mathbf{B}_n^0 = \mathbf{B}_n^+ \cap \mathbf{B}_n^-$. Then $\mathbf{B}_n^0$ is the subalgebra generated by $\mathbf{k}_i$, $-n \leqslant i \leqslant n-1$. We summarize the properties of these algebras; cf. [6, 8.2–3] and [12, 4.7–10].

For any $\lambda \in \Lambda([-n, n], r)$, let $\mathbf{h}_{\lambda,n} := \psi_n^{\geqslant 0}(\mathbf{k}'_\lambda) = \psi_n^{\leqslant 0}(\mathbf{k}'_\lambda)$.

Corollary 6.3. *Let $n \geqslant 1$.*

1. *Each of the following sets forms a basis for $\mathbf{B}_n^0$:*
   - (a) $\{\mathbf{h}_{\lambda,n} | \lambda \in \Lambda([-n, n], r)\}$*;*
   - (b) $\{\mathbf{k}^{\mathbf{j}} \mid \mathbf{j} \in \mathbb{N}^{[-n,n-1]}, \sigma(\mathbf{j}) \leqslant r\}$*.*
2. *For all $-n \leqslant i \leqslant n-1$ and $\lambda \in \Lambda([-n, n], r)$, $\mathbf{k}_i \mathbf{h}_{\lambda,n} = \upsilon^{\lambda_i} \mathbf{h}_{\lambda,n}$.*
3. $1 = \sum_{\lambda \in \Lambda([-n,n],r)} \mathbf{h}_{\lambda,n}$.
4. *The elements $\mathbf{h}_{\lambda,n}$ satisfy the commuting relations described in 5.4 with $\mathtt{k}_\lambda$ replaced by $\mathbf{h}_{\lambda,n}$ and $\mathtt{e}^{(A^+)}$, $\mathtt{f}^{(A^-)}$ by $\mathbf{e}^{(A^+)}$ and $\mathbf{f}^{(A^-)}$ for all $A \in \Xi^\pm([-n, n])$ and $\lambda \in \Lambda([-n, n], r)$.*

The elements $\mathbf{h}_{\lambda,n}$ play an important role in a construction of a spanning set for $\widetilde{\mathbf{V}}(\infty, r)$. Let

$$\mathcal{M}'_n = \{\mathbf{e}^{(A^+)} \mathbf{h}_{\lambda,n} \mathbf{f}^{(A^-)} \mid A \in \Xi^\pm([-n, n-1]), \lambda \in \Lambda([-n, n], r), \lambda \geqslant \boldsymbol{\sigma}(A)\},$$

where $\lambda \geqslant \boldsymbol{\sigma}(A)$ means $\lambda_i \geqslant \sigma_i(A)$ $\forall i$, and let $\mathcal{M}'_\infty = \cup_{n \geqslant 1} \mathcal{M}'_n$. Recall from 5.3 the set $\Lambda_{n,r,A}$.

Lemma 6.4. *For any $n \geqslant 1$, $A \in \Xi^\pm([-n, n-1])$, $\lambda \in \Lambda([-n, n], r)$, if $\lambda \notin \Lambda_{n,r,A}$, then there exist $f_{B,\mu}^{A,\lambda} \in \mathcal{Z}$ such that*

$$\mathbf{e}^{(A^+)} \mathbf{h}_{\lambda,n} \mathbf{f}^{(A^-)} = \sum_{\substack{B \in \Xi^\pm([-n,n-1]) \\ \mu \in \Lambda_{n,r,A}, \deg(B) < \deg(A)}} f_{B,\mu}^{A,\lambda} \mathbf{e}^{(B^+)} \mathbf{h}_{\mu,n} \mathbf{f}^{(B^-)}. \tag{6.4.1}$$

Hence, $\mathcal{M}'_\infty$ is a spanning set for $\widetilde{\mathbf{V}}(\infty, r)$.

Proof. We first observe, by 6.3(2)&(3), that all $\mathbf{k}_i$ and $\left[{\mathbf{k}_i;c \atop t} \right]$ are $\mathcal{Z}$-linear combinations of $\mathbf{h}_{\lambda,n}$ whenever $-n \leqslant i \leqslant n-1$. So (6.4.1) implies the assertion that $\mathcal{M}'_\infty$ spanns $\widetilde{\mathbf{V}}(\infty, r)$. We now prove (6.4.1).

Suppose $\lambda \in \Lambda([-n, n], r)$ and $\lambda_i < \sigma_i(A)$ where i is minimal. Then $\deg(A) \geqslant 1$. We apply induction on $\deg(A)$. If $\deg(A) = 1$, then $A^+ = E_{i-1,i}$ and $A^- = 0$ or $A^+ = 0$ and $A^- = E_{i,i-1}$.

By 5.4, we have $\mathbf{e}^{(A^+)}\mathbf{h}_{\lambda,n}\mathbf{f}^{(A^-)}=0$. Assume now $\deg(A)\geqslant 2$ and (6.4.1) is true for all A' with $\deg(A')<\deg(A)$. Let $A_i=(a^{(i)}_{k,l})$ be the submatrix of A such that $a^{(i)}_{k,l}=a_{k,l}$ if $k,l\leqslant i$ and $a^{(i)}_{k,l}=0$ otherwise. Since $A\in\Xi^{\pm}([-n,n-1])$, we can write $\mathbf{e}^{(A^+)}=\mathbf{m}_1\mathbf{e}^{(A_i^+)}$ and $\mathbf{f}^{(A^-)}=\mathbf{f}^{(A_i^-)}\mathbf{m}_1'$, where $\mathbf{m}_1$ is the monomial of $\mathbf{e}_j^{(a)}$ ($-n\leqslant j<n-1$, $a\geqslant 0$) and $\mathbf{m}_1'$ is the monomial of $\mathbf{f}_j^{(a)}$ ($-n\leqslant j<n-1$, $a\geqslant 0$). Then

$$\mathbf{e}^{(A^+)}\mathbf{h}_{\lambda,n}\mathbf{f}^{(A^-)}=\mathbf{m}_1\mathbf{e}^{(A_i^+)}\mathbf{h}_{\lambda,n}\mathbf{f}^{(A_i^-)}\mathbf{m}_1'.$$

Since $\lambda_j\geqslant\sigma_j(A_i^+)$ for all $j\leqslant i$, 6.3(4) implies

$$\mathbf{e}^{(A^+)}\mathbf{h}_{\lambda,n}\mathbf{f}^{(A^-)}=\mathbf{m}_1\mathbf{h}_{\lambda^*,n}\mathbf{e}^{(A_i^+)}\mathbf{f}^{(A_i^-)}\mathbf{m}_1',$$

where $\lambda^*\in\Lambda([-n,n],r)$ with $\lambda_i^*=\lambda-co(A_i^+)-ro(A_i^+)=\lambda_i-\sum_{j<i}a_{ji}=\lambda_i-\sigma_i(A^+)\geqslant 0$. On the other hand, the commutator formula 2.2 (see the remarks right after [12, 2.3]) implies

$$\mathbf{e}^{(A_i^+)}\mathbf{f}^{(A_i^-)}=\mathbf{f}^{(A_i^-)}\mathbf{e}^{(A_i^+)}+f,$$

where f is a $\mathcal{Z}$-linear combination of monomials $\mathbf{m}_j^{\mathbf{e}}\mathbf{h}_j\mathbf{m}_j^{\mathbf{f}}$ with $\deg(\mathbf{m}_j^{\mathbf{e}}\mathbf{m}_j^{\mathbf{f}})<\deg(A_i)$. Here, $\mathbf{m}_j^{\mathbf{e}}$ is a product of some $\mathbf{e}_s^{(a)}$, $\mathbf{m}_j^{\mathbf{f}}$ is a product of some $\mathbf{f}_s^{(b)}$, and $\mathbf{h}_j$ is a product of some $\left[\begin{smallmatrix}\widetilde{\mathbf{k}}_t;c\\ m\end{smallmatrix}\right]$ where[3] $-n\leqslant s,t\leqslant n-2$ and $a,b,c,m\in\mathbb{Z}$. Thus, $\deg(\mathbf{m}_1\mathbf{m}_j^{\mathbf{e}}\mathbf{m}_j^{\mathbf{f}}\mathbf{m}_1')<\deg(A)$. Now $\lambda_i<\sigma_i(A)$ implies $\lambda_i^*<\sigma_i(A^-)=\sigma_i(A_i^-)$. Hence $\mathbf{h}_{\lambda^*,n}\mathbf{f}^{(A_i^-)}=0$ by 6.3(4). Since $\mathbf{h}_j\in\mathbf{B}_n^0$ is a $\mathcal{Z}$-linear combination of $\mathbf{h}_{\mu,n}$ by 6.3(1-3), it follows from 6.3(4) that $\mathbf{e}^{(A^+)}\mathbf{h}_{\lambda,n}\mathbf{f}^{(A^-)}$ is a $\mathcal{Z}$-linear combination of $\mathbf{m}_1\mathbf{m}_j^{\mathbf{e}}\mathbf{h}_{\mu,n}\mathbf{m}_j^{\mathbf{f}}\mathbf{m}_1'$ with $\deg(\mathbf{m}_1\mathbf{m}_j^{\mathbf{e}}\mathbf{m}_j^{\mathbf{f}}\mathbf{m}_1')<\deg(A)$ and $\mu\in\Lambda([-n,n],r)$. Now, since $\widetilde{\mathbf{V}}(\infty,r)$ is a homomorphic image of $\mathbf{U}(\infty)$, 2.3 implies that each $\mathbf{m}_1\mathbf{m}_j^{\mathbf{e}}$ (resp., $\mathbf{m}_j^{\mathbf{f}}\mathbf{m}_1'$) is a $\mathcal{Z}$-linear combination of $\mathbf{e}^{(B)}$, $B\in\Xi^+([-n,n-1])$ (resp., $\mathbf{f}^{(B')}$, $B'\in\Xi^-([-n,n-1])$) with $\deg(B)=\deg(\mathbf{m}_1\mathbf{m}_j^{\mathbf{e}})$ (resp., $\deg(B')=\deg(\mathbf{m}_j^{\mathbf{f}}\mathbf{m}_1')$). Thus, each $\mathbf{m}_1\mathbf{m}_j^{\mathbf{e}}\mathbf{h}_{\mu,n}\mathbf{m}_j^{\mathbf{f}}\mathbf{m}_1'$ is a $\mathcal{Z}$-linear combination of $\mathbf{e}^{(A')^+}\mathbf{h}_{\mu,n}\mathbf{f}^{(A')^-}$ with $A'\in\Xi^{\pm}([-n,n-1])$ and $\deg(A')<\deg(A)$. Thus, our assertion follows from induction. $\square$

Though each $\mathcal{M}_n'$ is linearly independent as part of the integral basis for a q-Schur algebra (cf. (6.5.1) and 6.7 below), the set $\mathcal{M}_\infty'$ is unfortunately not linearly independent. In fact, we can see that $\mathrm{span}(\mathcal{M}_n')$ is a subspace of $\mathrm{span}(\mathcal{M}_{n+1}')$, but $\mathcal{M}_n'$ is not a subset of $\mathcal{M}_{n+1}'$. However, we shall use $\mathcal{M}_\infty'$ to construct several bases below.

Let

$$\widetilde{\mathcal{N}}_\infty:=\{\mathbf{e}^{(A^+)}\mathbf{k}^{\mathbf{j}}\mathbf{f}^{(A^-)}\mid A\in\Xi^{\pm}(\infty),\mathbf{j}\in\mathbb{N}^{\infty},\sigma(\mathbf{j})+\sigma(A)\leqslant r\}$$

and, for $n\geqslant 1$, let

$$\widetilde{\mathcal{N}}_n:=\{\mathbf{e}^{(A^+)}\mathbf{k}^{\mathbf{j}}\mathbf{f}^{(A^-)}\mid A\in\Xi^{\pm}([-n,n]),\mathbf{j}\in\Lambda'_{n,r,A}\},$$

where $\Lambda'_{n,r,A}$ is defined in Lemma 5.3. Then $\widetilde{\mathcal{N}}_n\subseteq\widetilde{\mathcal{N}}_{n+1}$ and $\widetilde{\mathcal{N}}_\infty=\bigcup_{n\geqslant 1}\widetilde{\mathcal{N}}_n$. Similarly, we can define $\mathcal{N}_\infty$ and $\mathcal{N}_n$ in the algebra $\mathbf{V}(\infty,r)$ so that $\mathcal{N}_\infty=\pi(\widetilde{\mathcal{N}}_\infty)$.

Corollary 6.5. *The set $\widetilde{\mathcal{N}}_\infty$ spans $\widetilde{\mathbf{V}}(\infty,r)$.*

Proof. By 6.4, it is enough to prove that, for any $n\geqslant 1$, the set $\mathcal{M}_n'$ lies in the span of $\widetilde{\mathcal{N}}_n$. In other words, we need to prove that, for any $A\in\Xi^{\pm}([-n,n-1])$, and $\lambda\in\Lambda([-n,n],r)$ with $\lambda\geqslant\boldsymbol{\sigma}(A)$, the element $\mathbf{e}^{(A^+)}\mathbf{h}_{\lambda,n}\mathbf{f}^{(A^-)}\in\mathrm{span}\,\widetilde{\mathcal{N}}_n$. We proceed by induction on $\deg(A)$; see (1.1.1). If $\deg(A)=0$, then by [12, 4.7] we have, for any $\lambda\in\Lambda([-n,n],r)$, $\mathbf{e}^{(A^+)}\mathbf{h}_{\lambda,n}\mathbf{f}^{(A^-)}=\mathbf{h}_{\lambda,n}$, which is, by 6.3(1), a

[3]Our assumption on A guarantees that $\mathbf{e}_{n-1}$ and $\mathbf{f}_{n-1}$ do not appear in $\mathbf{e}^{(A^+)}$ and $\mathbf{f}^{(A^-)}$. Thus, only the $\widetilde{\mathbf{k}}_t=\mathbf{k}_t\mathbf{k}_{t+1}^{-1}$ with $-n\leqslant t\leqslant n-2$ occur in the commutator formula; cf. 6.1(g).

linear combination of the elements $\mathbf{k}_{-n}^{j_{-n}}\cdots\mathbf{k}_{n-1}^{j_{n-1}}$ with $j_i\in\mathbb{N}$ and $j_{-n}+\cdots+j_{n-1}\leqslant r$, and hence, in the span of $\widetilde{\mathcal{N}}_n$. Assume now that $\deg(A)\geqslant 1$. By 6.3(1), we may write, for any $\mathbf{j}\in\mathbb{N}^{[-n,n]}$ with $j_n=0$,

$$\mathbf{e}^{(A^+)}\mathbf{k}^{\mathbf{j}}\mathbf{f}^{(A^-)}=\sum_{\mu\in\Lambda_{n,r,A}}v^{\mu\cdot\mathbf{j}}\mathbf{e}^{(A^+)}\mathbf{h}_{\mu,n}\mathbf{f}^{(A^-)}+\sum_{\nu\notin\Lambda_{n,r,A}}v^{\nu\cdot\mathbf{j}}\mathbf{e}^{(A^+)}\mathbf{h}_{\nu,n}\mathbf{f}^{(A^-)}.\tag{6.5.1}$$

By (6.4.1) and induction, the second sum is in the span of $\widetilde{\mathcal{N}}_n$. If $\mathbf{j}$ runs over the set $\Lambda'_{n,r,A}$ (see 5.3) then the determinant $\det(v^{\mu\cdot\mathbf{j}})_{\mu,\mathbf{j}}\neq 0$. Hence, $\mathbf{e}^{(A^+)}\mathbf{h}_{\mu,n}\mathbf{f}^{(A^-)}\in\operatorname{span}\widetilde{\mathcal{N}}_n$ for all $\mu\in\Lambda_{n,r,A}$. □

Theorem 6.6. *The algebra epimorphism $\pi:\widetilde{\mathbf{V}}(\infty,r)\to\mathbf{V}(\infty,r)$ defined in* (6.1.1) *is an isomorphism. In particular, if a coefficient $f_{B,\mu}^{A,\lambda}$ given in* (6.4.1) *is nonzero, then $B\prec A$.*

Proof. Since $\widetilde{\mathbf{V}}(\infty,r)$ is spanned by $\widetilde{\mathcal{N}}_\infty$ by 6.5, it is enough to prove that its image $\pi(\widetilde{\mathcal{N}}_\infty)=\mathcal{N}_\infty$ is linearly independent. Fix $n\geqslant 1$. By 5.2, for $A\in\Xi^\pm([-n,n])$, we have, for any $\mathbf{j}\in\mathbb{N}^{[-n,n-1]}$,

$$\mathtt{e}^{(A^+)}\mathtt{k}^{\mathbf{j}}\mathtt{f}^{(A^-)}=\sum_{\substack{\lambda\in\Lambda(\infty,r)\\ \forall j,\,\lambda_j\geqslant\sigma_j(A)}}v^{\lambda\cdot\mathbf{j}}\mathtt{e}^{(A^+)}\mathtt{k}_\lambda\mathtt{f}^{(A^-)}+\sum_{\substack{\mu\in\Lambda(\infty,r)\\ \exists j,\,\mu_j<\sigma_j(A)}}v^{\mu\cdot\mathbf{j}}\mathtt{e}^{(A^+)}\mathtt{k}_\mu\mathtt{f}^{(A^-)}.$$

Hence, by 5.6, there exist $f_{B,A}\in\mathbb{Q}(v)$ such that

$$\begin{aligned}\mathtt{e}^{(A^+)}\mathtt{k}^{\mathbf{j}}\mathtt{f}^{(A^-)}&=\sum_{\substack{\lambda\in\Lambda(\infty,r)\\ \forall i,\lambda_i\geqslant\sigma_i(A)}}v^{\lambda\cdot\mathbf{j}}\mathtt{e}^{(A^+)}\mathtt{k}_\lambda\mathtt{f}^{(A^-)}+\sum_{\substack{B\in\Xi(\infty,r)\\ B\prec A}}f_{B,A}\mathtt{m}^{(B)}\\ &=\sum_{\substack{\lambda\in\Lambda([-n,n],r)\\ \forall i,\lambda_i\geqslant\sigma_i(A)}}v^{\lambda\cdot\mathbf{j}}\mathtt{e}^{(A^+)}\mathtt{k}_\lambda\mathtt{f}^{(A^-)}+\sum_{\substack{\lambda\notin\Lambda([-n,n],r)\\ \forall i,\lambda_i\geqslant\sigma_i(A)}}v^{\lambda\cdot\mathbf{j}}\mathtt{e}^{(A^+)}\mathtt{k}_\lambda\mathtt{f}^{(A^-)}+\sum_{\substack{B\in\Xi(\infty,r)\\ B\prec A}}f_{B,A}\mathtt{m}^{(B)}.\end{aligned}$$

Since, for every $\lambda\in\Lambda([-n,n],r)$ with $\lambda_i\geqslant\sigma_i(A)$ $\forall i$,, $\mathtt{e}^{(A^+)}\mathtt{k}_\lambda\mathtt{f}^{(A^-)}=\mathtt{m}^{(A+D)}$ where D is a diagonal matrix whose ith entry is $\lambda_i-\sigma_i(A)$, it follows from 5.5 that the set $\{\mathtt{e}^{(A^+)}\mathtt{k}_\lambda\mathtt{f}^{(A^-)}\mid\lambda\in\Lambda_{n,r,A}\}$ is linearly independent. Hence, 5.3 implies that the set

$$\mathcal{N}_{n,A}=\{\mathtt{e}^{(A^+)}\mathtt{k}^{\mathbf{j}}\mathtt{f}^{(A^-)}\mid\mathbf{j}\in\mathbb{N}^{[-n,n]},j_n=0,\sigma(\mathbf{j})+\sigma(A)\leqslant r\}=\{\mathtt{e}^{(A^+)}\mathtt{k}^{\mathbf{j}}\mathtt{f}^{(A^-)}\mid\mathbf{j}\in\Lambda'_{n,r,A}\}$$

is linearly independent for each $A\in\Xi^\pm([-n,n])$. Note that 5.5 implies also that the union $\bigcup_{A\in\Xi^\pm([-n,n])}\{\mathtt{e}^{(A^+)}\mathtt{k}_\lambda\mathtt{f}^{(A^-)}\mid\lambda\in\Lambda_{n,r,A}\}$ is linearly independent. Thus, $\mathcal{N}_n=\cup_{A\in\Xi^\pm([-n,n])}\mathcal{N}_{n,A}$ is linearly independent. Since $\mathcal{N}_\infty=\bigcup_{n\geqslant1}\mathcal{N}_n$ and $\mathcal{N}_n\subseteq\mathcal{N}_{n+1}$, it follows that the set $\mathcal{N}_\infty$ is linearly independent.

The last assertion follows from 5.6 and 5.5. □

With the above result we will identify $\widetilde{\mathbf{V}}(\infty,r)$ with $\mathbf{V}(\infty,r)$. In particular we will identify $\mathbf{h}_{\lambda,n}$ with $\mathtt{h}_{\lambda,n}$, $\mathbf{e}_i$ with $\mathtt{e}_i$ etc. For $A\in\Xi^\pm(\infty)$ and $\mathbf{j}\in\mathbb{N}^\infty$, let

$$\mathtt{n}^{(A,\mathbf{j})}:=\mathtt{e}^{(A^+)}\mathtt{k}^{\mathbf{j}}\mathtt{f}^{(A^-)}.$$

Corollary 6.7. *The set $\mathcal{N}_\infty=\{\mathtt{n}^{(A,\mathbf{j})}\mid A\in\Xi^\pm(\infty),\mathbf{j}\in\mathbb{N}^\infty,\sigma(\mathbf{j})+\sigma(A)\leqslant r\}$ forms a basis for $\mathbf{V}(\infty,r)$. Moreover, if $\mathbf{j}\in\mathbb{N}^{[-n,n-1]}$ and $\mathbf{j}\notin\Lambda'_{n,r,A}$, then we have, for some $p_{B,\mathbf{j}'}\in\mathbb{Q}(v)$,*

$$\mathtt{e}^{(A^+)}\mathtt{k}^{\mathbf{j}}\mathtt{f}^{(A^-)}=\sum_{\substack{B\in\Xi^\pm([-n,n-1])\\ B\preccurlyeq A,\mathbf{j}'\in\Lambda'_{n,r,A}}}p_{B,\mathbf{j}'}\mathtt{n}^{(B,\mathbf{j}')}.$$

Proof. Using (6.5.1) and 6.4, the last assertion can be proved as the proof of [10, 6.7]. □

The basis $\mathcal{N}_\infty$ is the counterpart of the basis [10, 6.6(1)] for q-Schur algebras. The following is the counterpart of the basis given in [10, 6.6(2)].

Proposition 6.8. *The set*

$$\mathcal{B}_\infty = \{A(\mathbf{j}, r) \mid A \in \Xi^\pm(\infty), \mathbf{j} \in \mathbb{N}^\infty, \sigma(\mathbf{j}) + \sigma(A) \leqslant r\}$$

forms a basis for $\mathbf{V}(\infty, r)$.

Proof. By (5.5.2) and an inductive argument on $\|A\|$, we see that there exist $q_{B,\mathbf{j}} \in \mathbb{Q}(v)$ such that

$$A(\mathbf{j}, r) = v^a \mathtt{e}^{(A^+)}\mathtt{k}^{\mathbf{j}}\mathtt{f}^{(A^-)} + \sum_{\substack{B\in\Xi^\pm(\infty)\\ B\prec A, \mathbf{j}'\in\mathbb{N}^\infty}} q_{B,\mathbf{j}} \mathtt{e}^{(B^+)}\mathtt{k}^{\mathbf{j}'}\mathtt{f}^{(B^-)}.$$

Hence, the assertion follows from 6.7. □

We end this section with another application of the spanning set $\mathcal{M}'_\infty$ by displaying an integral monomial basis for the $\mathcal{Z}$-form of $\mathbf{V}(\infty, r)$; cf. [12, 5.4].

Let $V(\infty, r) = \xi_r(U(\infty))$. We further put

$$V^+(\infty, r) = \xi_r(U^+(\infty)), \quad V^-(\infty, r) = \xi_r(U^-(\infty)) \text{ and } V^0(\infty, r) = \xi_r(U^0(\infty)).$$

Lemma 6.9. (1) *The set* $\{\mathtt{e}^{(A)} \mid A \in \Xi^+(\infty), \sigma(A) \leqslant r\}$ (*resp.*, $\{\mathtt{f}^{(A)} \mid A \in \Xi^-(\infty)), \sigma(A) \leqslant r\}$) *forms a $\mathcal{Z}$-basis of* $V^+(\infty, r)$ (*resp.*, $V^-(\infty, r)$).

(2) *The set* $\mathcal{M}^0_\infty := \{\mathtt{k}_\lambda \mid \lambda \in \mathbb{N}^\infty, \sigma(\lambda) \leqslant r\}$ *forms a $\mathcal{Z}$-basis of* $V^0(\infty, r)$.

Proof. Let $A \in \Xi^+(\infty)$. If $\sigma(A) = \sum_i \sigma_i(A) > r$, then $\mathtt{e}^{(A)} = \sum_{\lambda\in\Lambda(\infty,r)} \mathtt{e}^{(A)}\mathtt{k}_\lambda = 0$ by 5.2 and 5.4. By 2.3 we have $V^+(\infty, r)$ is $\mathcal{Z}$ spanned by all $\mathtt{e}^{(A)}$ with $A \in \Xi^+(\infty)$ and $\sigma(A) \leqslant r$. Now (1) follows from 5.5. For $n \geqslant 1$ we let

$$\mathcal{M}^0_n := \Big\{ \prod_{-n\leqslant i\leqslant n-1} \begin{bmatrix} \mathtt{k}_i;0 \\ \mu_i \end{bmatrix} \mid \mu \in \Lambda([-n,n], r)\Big\} = \Big\{ \prod_{-n\leqslant i\leqslant n-1} \begin{bmatrix} \mathtt{k}_i;0 \\ \mu_i \end{bmatrix} \mid \mu \in \mathbb{N}^{[-n,n-1]}, \sigma(\mu) \leqslant r\Big\}.$$

We fix $n \geqslant 1$. For $\lambda, \mu \in \Lambda([-n, n], r)$ we write $\lambda \leqslant_n \mu$ if and only if $\lambda_i \leqslant \mu_i$ for $-n \leqslant i \leqslant n-1$. If $\lambda_i < \mu_i$ for some $-n \leqslant i \leqslant n-1$ then we write $\lambda <_n \mu$. For $\lambda \in \Lambda([-n, n], r)$ we have by 6.3(2&3)

$$\prod_{-n\leqslant i\leqslant n-1} \begin{bmatrix} \mathtt{k}_i;0 \\ \lambda_i \end{bmatrix} = \mathtt{h}_{\lambda,n} + \sum_{\mu\in\Lambda([-n,n],r),\lambda<_n\mu} \prod_{-n\leqslant i\leqslant n-1} \begin{bmatrix} \mu_i \\ \lambda_i \end{bmatrix} \mathtt{h}_{\mu,n}. \tag{6.9.1}$$

Hence, the set $\mathcal{M}^0_n$ is linearly independent and we have $\mathrm{span}_\mathcal{Z}\,\mathcal{M}^0_n = \mathrm{span}_\mathcal{Z}\{\mathtt{h}_{\lambda,n} | \lambda \in \Lambda([-n, n], r)\}$. Since $\mathcal{M}^0_n \subseteq \mathcal{M}^0_{n+1}$ and $\mathcal{M}^0_\infty = \bigcup_{n\geqslant1} \mathcal{M}^0_n$, the set $\mathcal{M}^0_\infty$ is linearly independent. It is clear that we have $V^0(\infty, r) = \mathrm{span}_\mathcal{Z}\{\mathtt{h}_{\lambda,n} | \lambda \in \Lambda([-n, n], r), n \geqslant 1\}$. Hence, we have $V^0(\infty, r) = \mathrm{span}_\mathcal{Z}\,\mathcal{M}^0_\infty$, proving (2). □

The proof above shows that every $\mathtt{h}_{\lambda,n} \in V(\infty, r)$. Hence, the set $\mathcal{M}'_\infty$ is a subset of the $\mathcal{Z}$-algebra $V(\infty, r)$. Thus, 6.4 implies immediately the following.

Corollary 6.10. *The set $\mathcal{M}'_\infty$ is a spanning set of the $\mathcal{Z}$-algebra* $V(\infty, r)$.

Proposition 6.11. *The set*

$$\mathcal{M}_\infty := \{\mathtt{e}^{(A^+)}\mathtt{k}_\lambda\mathtt{f}^{(A^-)} \mid A \in \Xi^\pm(\infty), \lambda \in \mathbb{N}^\infty, \lambda \geqslant \boldsymbol{\sigma}(A), \sigma(\lambda) \leqslant r\}$$

forms a $\mathcal{Z}$-basis for $V(\infty, r)$.

Proof. For any $n \geqslant 1$, let

$$\begin{aligned}\mathcal{M}_n := &\Big\{\mathtt{e}^{(A^+)}\prod_{-n\leqslant i\leqslant n-1}\begin{bmatrix}\mathtt{k}_i;0\\ \lambda_i\end{bmatrix}\mathtt{f}^{(A^-)}\big| A\in\Xi^{\pm}([-n,n-1]),\lambda\in\Lambda_{n,r,A}\Big\}\\ =&\Big\{\mathtt{e}^{(A^+)}\prod_{-n\leqslant i\leqslant n-1}\begin{bmatrix}\mathtt{k}_i;0\\ \lambda_i\end{bmatrix}\mathtt{f}^{(A^-)}\big| A\in\Xi^{\pm}([-n,n-1]),\lambda\in\mathbb{N}^{[-n,n-1]},\lambda\geqslant\boldsymbol{\sigma}(A),\sigma(\lambda)\leqslant r\Big\}.\end{aligned}$$

Then, $\mathcal{M}_n \subseteq \mathcal{M}_{n+1}$, $|\mathcal{M}_n| = |\mathcal{M}'_n|$ and $\mathcal{M}_\infty = \bigcup_{n\geqslant 1}\mathcal{M}_n$. Since, for $A \in \Xi^{\pm}([-n, n-1])$ and $\lambda \in \Lambda([-n, n], r)$ with $\lambda \geqslant \boldsymbol{\sigma}(A)$, (6.9.1) implies
(6.11.1)

$$\mathtt{e}^{(A^+)}\prod_{-n\leqslant i\leqslant n-1}\begin{bmatrix}\mathtt{k}_i;0\\ \lambda_i\end{bmatrix}\mathtt{f}^{(A^-)} = \mathtt{e}^{(A^+)}\mathtt{h}_{\lambda,n}\mathtt{f}^{(A^-)} + \sum_{\substack{\mu\in\Lambda([-n,n],r)\\ \lambda<_n\mu}}\prod_{-n\leqslant i\leqslant n-1}\begin{bmatrix}\mu_i\\ \lambda_i\end{bmatrix}\mathtt{e}^{(A^+)}\mathtt{h}_{\mu,n}\mathtt{f}^{(A^-)},$$

and $\mu >_n \lambda$ implies $\mu \geqslant \boldsymbol{\sigma}(A)$ (as $\sigma_i(A) = 0$ for $i \notin [-n, n-1]$), it follows that all $\mathtt{e}^{(A^+)}\mathtt{h}_{\mu,n}\mathtt{f}^{(A^-)} \in \mathcal{M}_n$ and $\mathrm{span}_{\mathcal{Z}}\,\mathcal{M}_n = \mathrm{span}_{\mathcal{Z}}\,\mathcal{M}'_n$. Consequently, $\mathrm{span}_{\mathcal{Z}}\,\mathcal{M}_\infty = \mathrm{span}_{\mathcal{Z}}\,\mathcal{M}'_\infty = V(\infty, r)$ by 6.10. On the other hand, the linear independence of $\mathcal{M}'_n$ implies that $\mathcal{M}_n$ is also linearly independent. Hence $\mathcal{M}_\infty$ is linearly independent. $\square$

Corollary 6.12. *(1) For any $n \geqslant 1$, the transition matrix between the bases $\mathcal{M}'_n$ and $\mathcal{M}_n$ is upper triangular with 1's on the diagonal.*

(2) Let $A \in \Xi^{\pm}(\infty)$ and $\lambda \in \mathbb{N}^\infty$ with $\sigma(\lambda) \leqslant r$. If $\lambda_i < \sigma_i(A)$ for some $i \in \mathbb{Z}$ then

$$\mathtt{e}^{(A^+)}\mathtt{k}_\lambda\mathtt{f}^{(A^-)} = \sum_{\substack{B\preccurlyeq A\\ \mu\geqslant\boldsymbol{\sigma}(A),\sigma(\mu)\leqslant r}} g_{B,\mu}\mathtt{e}^{(B^+)}\mathtt{k}_\mu\mathtt{f}^{(B^-)}\quad (g_{B,\mu}\in\mathcal{Z}, B\in\Xi^{\pm}(\infty),\mu\in\mathbb{N}^\infty)$$

Proof. (1) follows from (6.11.1). We now prove (2). Choose n such that $A \in \Xi^{\pm}([-n, n-1])$ and $\lambda \in \mathbb{N}^{[-n,n-1]}$. Then we have by (6.9.1)

$$\mathtt{e}^{(A^+)}\mathtt{k}_\lambda\mathtt{f}^{(A^-)} = \sum_{\mu\in\Lambda([-n,n],r)}\prod_{i\in\mathbb{Z}}\begin{bmatrix}\mu_i\\ \lambda_i\end{bmatrix}\mathtt{e}^{(A^+)}\mathtt{h}_{\mu,n}\mathtt{f}^{(A^-)}.$$

Now the result follows from (6.4.1) and (1). $\square$

7. Infinite q-Schur algebras

In this section, we will establish the isomorphism between $\mathbf{U}(\infty, r)$ and $\mathbf{V}(\infty, r)$ and discuss the relationship between $\mathbf{U}(\infty, r)$ and the infinite q-Schur algebra $\boldsymbol{\mathcal{S}}(\infty, r)$. We shall mainly work over the field $\mathbb{Q}(v)$ though results like 7.1 and 7.2 below continue to hold over $\mathcal{Z}$.

As in §2, let $\mathfrak{S}_r$ be the symmetric groups on r letters. For $\lambda \in \Lambda(\infty, r)$, let $\mathfrak{S}_\lambda$ be the Young subgroup of $\mathfrak{S}_r$, and let $\mathfrak{D}_\lambda$ be the set of distinguished right $\mathfrak{S}_\lambda$-coset representatives. Then, $\mathfrak{D}_{\lambda\mu} = \mathfrak{D}_\lambda \cap \mathfrak{D}_\mu^{-1}$ is the set of distinguished double $(\mathfrak{S}_\lambda, \mathfrak{S}_\mu)$-coset representatives. Let $\boldsymbol{\mathcal{H}}$ be the associated Hecke algebras over $\mathbb{Q}(v)$ with basis $\{T_w := v^{\ell(w)}\mathcal{T}_w\}_{w\in\mathfrak{S}_r}$, and let $x_\lambda = \sum_{w\in\mathfrak{S}_\lambda} T_w$.

Recall from §2 the $\boldsymbol{\mathcal{H}}$-module $\boldsymbol{\Omega}_\eta^{\otimes r}$. Clearly, there is an $\boldsymbol{\mathcal{H}}$-module embedding $i_n : \boldsymbol{\Omega}_{[-n,n]}^{\otimes r} \to \boldsymbol{\Omega}_{[-n-1,n+1]}^{\otimes r}$, and $\{(\boldsymbol{\Omega}_{[-n,n]}^{\otimes r}, i_n)\}_{n\geqslant 1}$ forms a direct system.

Lemma 7.1. *The $\boldsymbol{\mathcal{H}}$-module $\boldsymbol{\Omega}_\infty^{\otimes r}$ is the direct limit of $\boldsymbol{\mathcal{H}}$-modules $\{(\boldsymbol{\Omega}_{[-n,n]}^{\otimes r}, i_n)\}_{n\geqslant 1}$. Thus, $\boldsymbol{\mathcal{H}}$-module $\boldsymbol{\Omega}_\infty^{\otimes r}$ is isomorphic to the $\boldsymbol{\mathcal{H}}$-module $\oplus_{\lambda\in\Lambda(\infty,r)} x_\lambda\boldsymbol{\mathcal{H}}$.*

Proof. By [13, 5.1], we know there is a $\boldsymbol{\mathcal{H}}$-module isomorphism $\alpha_n : \boldsymbol{\Omega}^{\otimes r}_{[-n,n]} \to \oplus_{\lambda\in\Lambda([-n,n],r)} x_\lambda \boldsymbol{\mathcal{H}}$ and the following diagram are commutative.

$$\begin{array}{ccc}
\boldsymbol{\Omega}^{\otimes r}_{[-n,n]} & \xrightarrow{\alpha_n} & \bigoplus_{\lambda\in\Lambda([-n,n],r)} x_\lambda\boldsymbol{\mathcal{H}} \\
{\scriptstyle i_n}\Big\downarrow & & \Big\downarrow{\scriptstyle i'_n} \\
\boldsymbol{\Omega}^{\otimes r}_{[-n-1,n+1]} & \xrightarrow[\alpha_{n+1}]{} & \bigoplus_{\lambda\in\Lambda([-n-1,n+1],r)} x_\lambda\boldsymbol{\mathcal{H}},
\end{array}$$

where i_n and i'_n are natural injections. Since $\boldsymbol{\Omega}^{\otimes r}_\infty$ is the direct limit of $\{\boldsymbol{\Omega}^{\otimes r}_{[-n,n]}\}_{n\geqslant 1}$ and $\oplus_{\lambda\in\Lambda(\infty,r)} x_\lambda\boldsymbol{\mathcal{H}}$ is the direct limit of $\{\oplus_{\lambda\in\Lambda([-n,n],r)} x_\lambda\boldsymbol{\mathcal{H}}\}_{n\geqslant 1}$, the result follows. $\square$

Proposition 7.2. *We have the following algebra isomorphisms*

$$\boldsymbol{\mathcal{K}}(\infty,r) \overset{\theta}{\cong} \bigoplus_{\mu\in\Lambda(\infty,r)} \bigoplus_{\lambda\in\Lambda(\infty,r)} \mathrm{Hom}_{\boldsymbol{\mathcal{H}}}(x_\mu\boldsymbol{\mathcal{H}}, x_\lambda\boldsymbol{\mathcal{H}})$$

$$\boldsymbol{\mathcal{S}}(\infty,r) \overset{\tilde\theta}{\cong} \prod_{\mu\in\Lambda(\infty,r)} \bigoplus_{\lambda\in\Lambda(\infty,r)} \mathrm{Hom}_{\boldsymbol{\mathcal{H}}}(x_\mu\boldsymbol{\mathcal{H}}, x_\lambda\boldsymbol{\mathcal{H}}).$$

In particular, the algebra $\boldsymbol{\mathcal{K}}(\infty,r)$ can be regarded as a subalgebra (without the identity) of $\boldsymbol{\mathcal{S}}(\infty,r)$.

Proof. It is known (see, e.g., [9]) that, for any $n \geqslant 0$, the q-Schur algebras $\boldsymbol{\mathcal{K}}([-n,n],r)$ is isomorphic to $\bigoplus_{\lambda,\mu\in\Lambda([-n,n],r)} \mathrm{Hom}_{\boldsymbol{\mathcal{H}}}(x_\lambda\boldsymbol{\mathcal{H}}, x_\mu\boldsymbol{\mathcal{H}})$. Since $\boldsymbol{\mathcal{K}}(\infty,r)$ is the direct limit of $\{\boldsymbol{\mathcal{K}}([-n,n],r)\}_{n\geqslant 1}$, the first assertion follows. By 7.1, we have

$$\boldsymbol{\mathcal{S}}(\infty,r) \cong \prod_{\mu\in\Lambda(\infty,r)} \mathrm{Hom}_{\boldsymbol{\mathcal{H}}}(x_\mu\boldsymbol{\mathcal{H}}, \bigoplus_{\lambda\in\Lambda(\infty,r)} x_\lambda\boldsymbol{\mathcal{H}}).$$

Now, one checks easily that, for each $\mu\in\Lambda(\infty,r)$, there is a natural isomorphism

$$\bigoplus_{\lambda\in\Lambda(\infty,r)} \mathrm{Hom}_{\boldsymbol{\mathcal{H}}}(x_\mu\boldsymbol{\mathcal{H}}, x_\lambda\boldsymbol{\mathcal{H}}) \cong \mathrm{Hom}_{\boldsymbol{\mathcal{H}}}(x_\mu\boldsymbol{\mathcal{H}}, \bigoplus_{\lambda\in\Lambda(\infty,r)} x_\lambda\boldsymbol{\mathcal{H}}),$$

which induces the required algebra isomorphism. $\square$

The isomorphism θ can be made explicit. For $\lambda\in\Lambda(\infty,r)$, let $Y(\lambda)$ be the Young diagram of λ which is a collection of boxes, arranged in left justified rows with λ_i boxes in row i for all $i\in\mathbb{Z}$, and let t^λ be the λ-tableau in which the numbers $1, 2\cdots, r$ appear in order from left to right down successive (non-empty) rows of $Y(\lambda)$. Let R^λ_i ($i\in\mathbb{Z}$) be the set of entries in the i-th row of t^λ. Clearly, $R^\lambda_i \neq \emptyset$ if and only if $\lambda_i > 0$. Let

$$\mathfrak{D}(\infty,r) = \{(\lambda,d,\mu) \mid \lambda,\mu\in\Lambda(\infty,r),\ d\in\mathfrak{D}_{\lambda\mu}\}. \tag{7.2.1}$$

By [15, (1.3.10)], every element $(\lambda,d,\mu)\in\mathfrak{D}(\infty,r)$ defines a matrix $A=(a_{i,j})_{i,j\in\mathbb{Z}}\in\Xi(\infty,r)$ such that $a_{i,j} = |R^\lambda_i \cap dR^\mu_j|$. This defines a bijective map

$$\jmath : \mathfrak{D}(\infty,r) \longrightarrow \Xi(\infty,r).$$

Define, for any $\lambda,\mu\in\Lambda(\infty,r)$ and $w\in\mathfrak{D}_{\lambda,\mu}$, a map

$$\phi^w_{\lambda\mu} : \oplus_{\lambda\in\Lambda(\infty,r)} x_\lambda\mathcal{H} \to \oplus_{\lambda\in\Lambda(\infty,r)} x_\lambda\mathcal{H} \tag{7.2.2}$$

by setting $\phi^w_{\lambda\mu}(x_\nu h) = \delta_{\mu,\nu}\sum_{x\in\mathfrak{S}_\lambda w\mathfrak{S}_\mu} h$. Then, the set $\{\phi^w_{\lambda\mu} \mid (\lambda,d,\mu)\in\mathfrak{D}(\infty,r)\}$ forms a basis for $\bigoplus_{\mu\in\Lambda(\infty,r)}\bigoplus_{\lambda\in\Lambda(\infty,r)} \mathrm{Hom}_{\boldsymbol{\mathcal{H}}}(x_\mu\boldsymbol{\mathcal{H}}, x_\lambda\boldsymbol{\mathcal{H}})$. By [9, 1.4], the isomorphism θ is induced by $\jmath$. In other words, if $\jmath(\lambda,w,\mu)=A$, then $\theta(e_A)=\phi^w_{\lambda\mu}$. In particular, by 5.1, we have $\theta(\mathtt{k}_\lambda)=\phi^1_{\lambda\lambda}$.

We now identify $\boldsymbol{\mathcal{S}}(\infty,r)$ as $\prod_{\mu\in\Lambda(\infty,r)}\bigoplus_{\lambda\in\Lambda(\infty,r)}\mathrm{Hom}_{\boldsymbol{\mathcal{H}}}(x_\mu\boldsymbol{\mathcal{H}},x_\lambda\boldsymbol{\mathcal{H}})$ under $\tilde\theta$. Thus, the elements of $\boldsymbol{\mathcal{S}}(\infty,r)$ have the form $\left(\sum_{\lambda\in\Lambda(\infty,r)}f_{\lambda,\mu}\right)_{\mu\in\Lambda(\infty,r)}$, where $f_{\lambda,\mu}\in\mathrm{Hom}_{\boldsymbol{\mathcal{H}}}(x_\mu\boldsymbol{\mathcal{H}},x_\lambda\boldsymbol{\mathcal{H}})$ for all $\lambda,\mu\in\Lambda(\infty,r)$, which represents the map given by

$$\left(\sum_{\lambda\in\Lambda(\infty,r)}f_{\lambda,\mu}\right)_{\mu\in\Lambda(\infty,r)}\big(\sum_{\nu\in\Lambda(\infty,r)}a_\nu\big)=\sum_{\lambda,\mu\in\Lambda(\infty,r)}f_{\lambda,\mu}(a_\mu),$$

where $\sum_\nu a_\nu\in\oplus_\lambda x_\lambda\boldsymbol{\mathcal{H}}$. This identification allows us to regard the infinite q-Schur algebra $\boldsymbol{\mathcal{S}}(\infty,r)$ as the †-completion of $\boldsymbol{\mathcal{K}}(\infty,r)$.

Proposition 7.3. *Let $\widehat{\boldsymbol{\mathcal{K}}}^\dagger(\infty,r)$ be the completion algebra of $\boldsymbol{\mathcal{K}}(\infty,r)$ constructed as in 1.1. Then there is an algebra isomorphism*

$$\varsigma_r:\widehat{\boldsymbol{\mathcal{K}}}^\dagger(\infty,r)\xrightarrow{\sim}\boldsymbol{\mathcal{S}}(\infty,r),\quad\sum_{A\in\Xi(\infty,r)}\beta_A[A]\longmapsto\Bigg(\sum_{\substack{co(A)=\mu\\A\in\Xi(\infty,r)}}\beta_A[A]\Bigg)_{\mu\in\Lambda(\infty,r)}.$$

Proof. Clearly, ς_r is induced by the natural monomorphism $\varsigma_r:\boldsymbol{\mathcal{K}}(\infty,r)\to\boldsymbol{\mathcal{S}}(\infty,r)$ given in 7.2. □

By 1.1, there is another completion algebra $\widehat{\boldsymbol{\mathcal{K}}}(\infty,r)$ which is a subalgebra of $\widehat{\boldsymbol{\mathcal{K}}}^\dagger(\infty,r)$ and which contains $\mathbf{V}(\infty,r)$ as a subalgebras. Thus, restriction gives an algebra monomorphism $\varsigma_r:\mathbf{V}(\infty,r)\to\boldsymbol{\mathcal{S}}(\infty,r)$.

We are now ready to establish the isomorphism $\mathbf{U}(\infty,r)\to\mathbf{V}(\infty,r)$.

There is a similar basis $\{\phi^d_{\lambda\mu}\mid\lambda,\mu\in\Lambda([-n,n],r),\ d\in\mathfrak{D}_{\lambda\mu}\}$ for $\boldsymbol{\mathcal{S}}([-n,n],r)$ by [5], and $[A]=\upsilon^{-d_A}\phi^d_{\lambda\mu}$, where $A=\jmath(\lambda,d,\mu)$, if we identify $\mathbf{U}([-n,n],r)$ with $\boldsymbol{\mathcal{S}}([-n,n],r)$ (cf. [8, A.1]).

Theorem 7.4. *The following diagram is commutative*

$$\begin{array}{ccc}\mathbf{V}(\infty) & \xrightarrow{\xi_r} & \mathbf{V}(\infty,r)\\ \sim\downarrow & & \downarrow\varsigma_r\\ \mathbf{U}(\infty) & \xrightarrow[\zeta_r]{} & \boldsymbol{\mathcal{S}}(\infty,r)\end{array}$$

Hence, we have an isomorphism $\mathbf{U}(\infty,r)\cong\mathbf{V}(\infty,r)$. Moreover, the map $\zeta_r:\mathbf{U}(\infty)\to\boldsymbol{\mathcal{S}}(\infty,r)$ is not surjective for any $r\geqslant 1$.

Proof. Recall from 4.3 that we may identify $\mathbf{U}(\infty)$ with $\mathbf{V}(\infty)$. Fix some $i\in\mathbb{Z}$. For $\omega\in\boldsymbol{\Omega}_\infty^{\otimes r}$ we choose $n\geqslant 1$ such that $-n\leqslant i<n$ and $\omega\in\boldsymbol{\Omega}^{\otimes r}_{[-n,n]}$. Then $E_i\cdot\omega=E_{i,i+1}(\mathbf{0},r)_{[-n,n]}\cdot\omega$ by a result for q-Schur algebras. It is clear that we have $E_{i,i+1}(\mathbf{0},r)_{[-n,n]}\cdot\omega=E_{i,i+1}(\mathbf{0},r)_\infty\cdot\omega$. Hence $E_i\cdot\omega=E_{i,i+1}(\mathbf{0},r)_\infty\cdot\omega$ for any $\omega\in\boldsymbol{\Omega}_\infty^{\otimes r}$. Similarly, we have $F_i\cdot\omega=E_{i+1,i}(\mathbf{0},r)_\infty\cdot\omega$ and $K_i\cdot\omega=0(\boldsymbol{e}_i,r)_\infty\cdot\omega$ for any $\omega\in\boldsymbol{\Omega}_\infty^{\otimes r}$. The first assertion follows.

To see the last statement, it suffices to prove that the injective map $\varsigma_r:\widehat{\boldsymbol{\mathcal{K}}}(\infty,r)\to\boldsymbol{\mathcal{S}}(\infty,r)$ is not surjective. Fix a $\lambda\in\Lambda(\infty,r)$. For any $\mu\in\Lambda(\infty,r)$, construct a matrix $A_\mu\in\Xi(\infty,r)$ such that $ro(A_\mu)=\lambda$ and $co(A_\mu)=\mu$. Clearly, the element $([A_\mu])_{\mu\in\Lambda(\infty,r)}$ belongs to $\boldsymbol{\mathcal{S}}(\infty,r)$, but not in $\widehat{\boldsymbol{\mathcal{K}}}(\infty,r)$. □

With the above result, we will identify $\mathbf{U}(\infty,r)$ with $\mathbf{V}(\infty,r)$. Thus, the algebra homomorphism $\zeta_r:\mathbf{U}(\infty)\to\boldsymbol{\mathcal{S}}(\infty,r)$ sends E_i,K_i,F_i to $\mathtt{e}_i,\mathtt{k}_i,\mathtt{f}_i$, respectively. Now, by applying ζ_r to the graded components in (2.1.2), 5.4 implies immediately the following.

Corollary 7.5. *Let $\lambda \in \Lambda(\infty, r)$ and $\nu', \nu'' \in \mathbb{Z}\Pi(\infty)$. If $t' \in \mathbf{U}(\infty)_{\nu'}$ and $t'' \in \mathbf{U}(\infty)_{\nu''}$, then*

$$\zeta_r(t')[\mathrm{diag}(\lambda)] = \begin{cases} [\mathrm{diag}(\lambda+\nu')]\zeta_r(t') & \text{if } \lambda+\nu' \in \Lambda(\infty,r); \\ 0 & \text{otherwise,} \end{cases}$$

and

$$[\mathrm{diag}(\lambda)]\zeta_r(t'') = \begin{cases} \zeta_r(t'')[\mathrm{diag}(\lambda-\nu'')] & \text{if } \lambda-\nu'' \in \Lambda(\infty,r); \\ 0 & \text{otherwise.} \end{cases}$$

8. Modified quantum $\mathfrak{gl}_\infty$ and related algebras

The interpretation of the infinite q-Schur algebra $\boldsymbol{\mathcal{S}}(\infty, r)$ as the †-completion $\widehat{\boldsymbol{\mathcal{K}}}^\dagger(\infty, r)$ of $\boldsymbol{\mathcal{K}}(\infty, r)$ suggests us to introduce the †-completion $\widehat{\boldsymbol{\mathcal{K}}}^\dagger(\infty)$ of the algebra $\boldsymbol{\mathcal{K}}(\infty)$ which contains $\widehat{\boldsymbol{\mathcal{K}}}(\infty)$, and hence $\mathbf{U}(\infty)$, as a subalgebra. Thus, it is natural to expect that the map $\zeta_r : \mathbf{U}(\infty) \to \boldsymbol{\mathcal{S}}(\infty, r)$ given in (2.3.1) may be extended to an epimorphism from $\widehat{\boldsymbol{\mathcal{K}}}^\dagger(\infty)$ to $\boldsymbol{\mathcal{S}}(\infty, r)$. In this section, we will establish this epimorphism through an investigation of the epimorphism $\dot{\zeta}_r : \boldsymbol{\mathcal{K}}(\infty) \to \boldsymbol{\mathcal{K}}(\infty, r)$ induced by ζ_r via the modified quantum group $\dot{\mathbf{U}}(\infty)$ of $\mathbf{U}(\infty)$ (see [24, 23.1]).

Recall from Theorem 4.3 that $\mathbf{U}(\infty)$ has a basis $\{A(\mathbf{j})\}_{A,\mathbf{j}}$. Thus, for any $\lambda, \mu \in \mathbb{Z}^\infty$, there is a linear map from $\mathbf{U}(\infty)$ to $\boldsymbol{\mathcal{K}}(\infty)$ sending u to $[\mathrm{diag}(\lambda)]u[\mathrm{diag}(\mu)]$. Let

$${}_\lambda\mathbf{K}_\mu = \sum_{\mathbf{j}\in\mathbb{Z}^\infty} (K^{\mathbf{j}} - \upsilon^{\lambda\cdot\mathbf{j}})\mathbf{U}(\infty) + \sum_{\mathbf{j}\in\mathbb{Z}^\infty} \mathbf{U}(\infty)(K^{\mathbf{j}} - \upsilon^{\mu\cdot\mathbf{j}}) \text{ and } {}_\lambda\overline{\mathbf{U}(\infty)}_\mu := \mathbf{U}(\infty)/{}_\lambda\mathbf{K}_\mu.$$

Since $[\mathrm{diag}(\lambda)]{}_\lambda\mathbf{K}_\mu[\mathrm{diag}(\mu)] = 0$, this map induces a linear map ${}_\lambda\overline{\mathbf{U}(\infty)}_\mu \to \boldsymbol{\mathcal{K}}(\infty)$ sending $\pi_{\lambda\mu}(u)$ to $[\mathrm{diag}(\lambda)]u[\mathrm{diag}(\mu)]$, where $\pi_{\lambda\mu} : \mathbf{U}(\infty) \to {}_\lambda\overline{\mathbf{U}(\infty)}_\mu$ is the canonical projection. Thus, we obtain a linear map

$$f : \dot{\mathbf{U}}(\infty) := \bigoplus_{\lambda,\mu\in\mathbb{Z}^\infty} {}_\lambda\overline{\mathbf{U}(\infty)}_\mu \longrightarrow \boldsymbol{\mathcal{K}}(\infty).$$

We will introduce a multiplication in $\dot{\mathbf{U}}(\infty)$ and prove that f is an algebra isomorphism. We need some preparation.

Lemma 8.1. *Let $A \in \Xi^\pm(\infty)$ and $\lambda, \mu \in \mathbb{Z}^\infty$. If $\lambda - \mu \neq ro(A) - co(A)$, then we have $\pi_{\lambda\mu}(A(\mathbf{j})) = 0$ for any $\mathbf{j} \in \mathbb{Z}^\infty$.*

Proof. Since $\mu - co(A) \neq \lambda - ro(A)$, there exist $\mathbf{j}' \in \mathbb{Z}^\infty$ such that $\mathbf{j}' \cdot (\mu - co(A)) \neq \mathbf{j}' \cdot (\lambda - ro(A))$. By (4.2.1) we have for any $\mathbf{j} \in \mathbb{Z}^\infty$

$$(\upsilon^{\mathbf{j}'\cdot(\mu-co(A))} - \upsilon^{\mathbf{j}'\cdot(\lambda-ro(A))})A(\mathbf{j}) = \upsilon^{-\mathbf{j}'\cdot ro(A)}(K^{\mathbf{j}'} - \upsilon^{\lambda\cdot\mathbf{j}'})A(\mathbf{j}) - \upsilon^{-\mathbf{j}'\cdot co(A)}A(\mathbf{j})(K^{\mathbf{j}'} - \upsilon^{\mu\cdot\mathbf{j}'}).$$

The assertion follows. □

Recall the algebra grading of $\mathbf{U}(\infty) = \bigoplus_{\nu\in\mathbb{Z}\Pi(\infty)} \mathbf{U}(\infty)_\nu$ given in (2.1.2).

Lemma 8.2. *For $\lambda, \mu \in \mathbb{Z}^\infty$ and $\nu \in \mathbb{Z}\Pi(\infty)$. If $\nu \neq \lambda - \mu$ then we have $\pi_{\lambda\mu}(\mathbf{U}(\infty)_\nu) = 0$. Hence, ${}_\lambda\overline{\mathbf{U}(\infty)}_\mu = \pi_{\lambda\mu}(\mathbf{U}(\infty)_{\lambda-\mu})$.*

Proof. Let $t \in \mathbf{U}(\infty)_\nu$. Since $K^{\mathbf{j}}t = \upsilon^{\nu\cdot\mathbf{j}}tK^{\mathbf{j}}$ for $\mathbf{j} \in \mathbb{Z}^\infty$, we have $\upsilon^{\lambda\cdot\mathbf{j}}\pi_{\lambda,\mu}(t) = \pi_{\lambda\mu}(K^{\mathbf{j}}t) = \pi_{\lambda\mu}(\upsilon^{\nu\cdot\mathbf{j}}tK^{\mathbf{j}}) = \upsilon^{\mu\cdot\mathbf{j}+\nu\cdot\mathbf{j}}\pi_{\lambda\mu}(t)$ for all $\mathbf{j} \in \mathbb{Z}^\infty$. Since $\nu \neq \lambda - \mu$ there exist $\mathbf{j} \in \mathbb{Z}^\infty$ such that $\nu \cdot \mathbf{j} \neq (\lambda - \mu) \cdot \mathbf{j}$. Hence $\pi_{\lambda\mu}(t) = 0$. □

For any $\lambda',\mu',\lambda'',\mu''\in\mathbb{Z}^\infty$ with $\lambda'-\mu',\lambda''-\mu''\in\mathbb{Z}\Pi(\infty)$ and any $t\in\mathbf{U}(\infty)_{\lambda'-\mu'}$, $s\in\mathbf{U}(\infty)_{\lambda''-\mu''}$, define the product in $\dot{\mathbf{U}}(\infty)$

$$\pi_{\lambda'\mu'}(t)\pi_{\lambda''\mu''}(s)=\begin{cases}\pi_{\lambda'\mu''}(ts), & \text{if } \mu'=\lambda''\\ 0 & \text{otherwise.}\end{cases}$$

Using 8.2 one can easily check the above product defines an associative $\mathbb{Q}(v)$-algebra structure on $\dot{\mathbf{U}}(\infty)$.

Theorem 8.3. *The linear map $f:\dot{\mathbf{U}}(\infty)\to\boldsymbol{\mathcal{K}}(\infty)$ sending $\pi_{\lambda\mu}(u)$ to $[\mathrm{diag}(\lambda)]u[\mathrm{diag}(\mu)]$ for all $u\in\mathbf{U}(\infty)$ and $\lambda,\mu\in\mathbb{Z}^\infty$, is an algebra isomorphism.*

Proof. Since, for any $A\in\widetilde{\Xi}^\pm(\infty)$, $\mathbf{j}\in\mathbb{Z}^\infty$ and $\lambda,\mu\in\mathbb{Z}^\infty$ with $\lambda-\mu=ro(A)-co(A)$, we have

(8.3.1) $[\mathrm{diag}(\lambda)]A(\mathbf{j})[\mathrm{diag}(\mu)]=v^{(\lambda-ro(A))\cdot\mathbf{j}}[A+\mathrm{diag}(\lambda-ro(A))]=v^{(\mu-co(A))\cdot\mathbf{j}}[A+\mathrm{diag}(\mu-co(A))]$,

it follows that $[A]=[\mathrm{diag}(ro(A))]A^\pm(\mathbf{0})[\mathrm{diag}(co(A))]$ for all $A\in\widetilde{\Xi}(\infty)$, and so

$$\boldsymbol{\mathcal{K}}(\infty)=\bigoplus_{\lambda,\mu\in\mathbb{Z}^\infty}[\mathrm{diag}(\lambda)]\mathbf{U}(\infty)[\mathrm{diag}(\mu)].$$

So f is surjective. Suppose now $\sum_{\lambda,\mu}\pi_{\lambda\mu}(u)\in\ker(f)$, where $u=\sum_{\substack{A\in\Xi^\pm(\infty)\\ \mathbf{j}\in\mathbb{Z}^\infty}}\beta_{A,\mathbf{j}}A(\mathbf{j})$ with $\beta_{A,\mathbf{j}}\in\mathbb{Q}(v)$. Then, $[\mathrm{diag}(\lambda)]u[\mathrm{diag}(\mu)]=0$ for all λ,μ. Let

$$u_{\lambda\mu}=\sum_{\substack{A\in\Xi^\pm(\infty),\mathbf{j}\in\mathbb{Z}^\infty\\ \lambda-\mu=ro(A)-co(A)}}\beta_{A,\mathbf{j}}A(\mathbf{j}).$$

By 8.1 we have $\pi_{\lambda\mu}(u)=\pi_{\lambda\mu}(u_{\lambda\mu})$. On the other hand, (8.3.1) implies

$$0=[\mathrm{diag}(\lambda)]u_{\lambda\mu}[\mathrm{diag}(\mu)]=\sum_{\substack{A\in\Xi^\pm(\infty)\\ \lambda-\mu=ro(A)-co(A)}}\Bigg(\sum_{\mathbf{j}\in\mathbb{Z}^\infty}v^{(\lambda-ro(A))\cdot\mathbf{j}}\beta_{A,\mathbf{j}}\Bigg)[A+\mathrm{diag}(\lambda)-ro(A)].$$

Hence, $\sum_{\mathbf{j}\in\mathbb{Z}^\infty}v^{(\lambda-ro(A))\cdot\mathbf{j}}\beta_{A,\mathbf{j}}=0$ for any $A\in\Xi^\pm(\infty)$ with $\lambda-\mu=ro(A)-co(A)$. By (4.2.1) we obtain

$$\begin{aligned}u_{\lambda\mu}&=\sum_{\substack{A\in\Xi^\pm(\infty),\mathbf{j}\in\mathbb{Z}^\infty\\ \lambda-\mu=ro(A)-co(A)}}\beta_{A,\mathbf{j}}A(\mathbf{j})-\sum_{\substack{A\in\Xi^\pm(\infty),\mathbf{j}\in\mathbb{Z}^\infty\\ \lambda-\mu=ro(A)-co(A)}}v^{(\lambda-ro(A))\cdot\mathbf{j}}\beta_{A,\mathbf{j}}A(0)\\ &=\sum_{\substack{A\in\Xi^\pm(\infty),\mathbf{j}\in\mathbb{Z}^\infty\\ \lambda-\mu=ro(A)-co(A)}}(K^{\mathbf{j}}-v^{\lambda\cdot\mathbf{j}})v^{-\mathbf{j}\cdot ro(A)}\beta_{A,\mathbf{j}}A(0)\in{}_\lambda\mathbf{K}_\mu.\end{aligned}$$

Hence $\pi_{\lambda\mu}(u)=\pi_{\lambda\mu}(u_{\lambda\mu})=0$ for all λ,μ. So f is injective.

We now prove f is an algebra homomorphism. Let $u_1,u_2\in\mathbf{U}(\infty)$ and $\lambda',\mu',\lambda'',\mu''\in\mathbb{Z}^\infty$. If $\mu'\neq\lambda''$ or $\lambda'-\mu'\notin\mathbb{Z}\Pi(\infty)$ or $\lambda''-\mu''\notin\mathbb{Z}\Pi(\infty)$ then by definition and 8.2

$$f(\pi_{\lambda'\mu'}(u_1)\pi_{\lambda''\mu''}(u_2))=0=f(\pi_{\lambda'\mu'}(u_1))f(\pi_{\lambda''\mu''}(u_2)).$$

It remains to prove the case when $\mu'=\lambda''$, $\lambda'-\mu'\in\mathbb{Z}\Pi(\infty)$ and $\lambda''-\mu''\in\mathbb{Z}\Pi(\infty)$. By 8.2, we may assume $u_1\in\mathbf{U}(\infty)_{\lambda'-\mu'}$ and $u_2\in\mathbf{U}(\infty)_{\lambda''-\mu''}$. Observe that, for $u=E^{(A^+)}K^{\mathbf{j}}F^{(A^-)}\in\mathbf{U}(\infty)_{\lambda-\mu}$, where $\lambda,\mu\in\mathbb{Z}^\infty$ with $\lambda-\mu\in\mathbb{Z}\Pi(\infty)$, since $\lambda-\mu=\sum_{i\leqslant h<j}(a_{ij}-a_{ji})\alpha_h$, it follows from 4.1 that

$$[\mathrm{diag}(\lambda)]u[\mathrm{diag}(\mu)]=[\mathrm{diag}(\lambda)][\mathrm{diag}(\mu+\sum_{i\leqslant k<j}(a_{ij}-a_{ji})\alpha_k)]u=[\mathrm{diag}(\lambda)]u.$$

Hence, by 2.3, $[\mathrm{diag}(\lambda)]u'[\mathrm{diag}(\mu)] = [\mathrm{diag}(\lambda)]u'$ for any $u' \in \mathbf{U}(\infty)_{\lambda-\mu}$. Thus, if $\mu' = \lambda''$, then

$$\begin{aligned} f(\pi_{\lambda',\mu'}(u_1)\pi_{\mu',\mu''}(u_2)) &= f(\pi_{\lambda'\mu''}(u_1u_2)) = [\mathrm{diag}(\lambda')]u_1u_2[\mathrm{diag}(\mu'')] \\ &= ([\mathrm{diag}(\lambda')]u_1[\mathrm{diag}(\mu')])u_2[\mathrm{diag}(\mu'')] \\ &= f(\pi_{\lambda',\mu'}(u_1))f(\pi_{\mu',\mu''}(u_2)). \end{aligned}$$

as required. □

With the above result, we shall identify $\dot{\mathbf{U}}(\infty)$ with $\boldsymbol{\mathcal{K}}(\infty)$. In particular, we shall identify $\pi_{\lambda\mu}(u)$ with $[\mathrm{diag}(\lambda)]u[\mathrm{diag}(\mu)]$ for $u \in \mathbf{U}(\infty)$ and $\lambda, \mu \in \mathbb{Z}^{\infty}$.

Since both algebras $\boldsymbol{\mathcal{K}}(\infty) = \dot{\mathbf{U}}(\infty)$ and $\mathbf{V}(\infty) = \mathbf{U}(\infty)$ are subalgebras of $\widehat{\boldsymbol{\mathcal{K}}}(\infty)$, the definition of $\mathbf{V}(\infty)$ implies that the algebra $\boldsymbol{\mathcal{K}}(\infty)$ is a $\mathbf{U}(\infty)$-bimodule with an action induced by multiplication. We now have the following interpretation of this bimodule structure.

Corollary 8.4. *The algebra $\mathbf{U}(\infty)$ acts on $\dot{\mathbf{U}}(\infty)$ by the following rules:*

$$t'\pi_{\lambda\mu}(s)t'' = \pi_{\lambda+\nu',\mu-\nu''}(t'st'')$$

for all $t' \in \mathbf{U}(\infty)(\nu')$, $t'' \in \mathbf{U}(\infty)(\nu'')$ and $s \in \mathbf{U}(\infty)$, where $\lambda, \mu \in \mathbb{Z}^{\infty}$ and $\nu', \nu'' \in \mathbb{Z}\Pi(\infty)$.

Proof. By 4.1 and 2.3 we have $t[\mathrm{diag}(\lambda)] = [\mathrm{diag}(\lambda+\nu)]t$ and $[\mathrm{diag}(\lambda)]t = t[\mathrm{diag}(\lambda-\nu)]$ for $t \in \mathbf{U}(\infty)_\nu$. Hence we have $t'\pi_{\lambda\mu}(s)t'' = t'[\mathrm{diag}(\lambda)]s[\mathrm{diag}(\mu)]t'' = [\mathrm{diag}(\lambda+\nu')]t'st''[\mathrm{diag}(\mu+\nu'')] = \pi_{\lambda+\nu',\mu-\nu''}(t'st'')$. □

Similar to [25, 3.4], we define a map $\dot{\zeta}_r$ from $\dot{\mathbf{U}}(\infty)$ to $\mathbf{U}(\infty, r)$ as follows.

Theorem 8.5. *The map $\dot{\zeta}_r : \dot{\mathbf{U}}(\infty) \to \mathbf{U}(\infty, r)$ defined by*

$$\dot{\zeta}_r(\pi_{\lambda\mu}(u)) = \begin{cases} [\mathrm{diag}(\lambda)]\zeta_r(u)[\mathrm{diag}(\mu)] & \text{if } \lambda, \mu \in \Lambda(\infty, r); \\ 0 & \text{otherwise} \end{cases}$$

for $u \in \mathbf{U}(\infty)$ and $\lambda, \mu \in \mathbb{Z}^{\infty}$ is an algebra homomorphism.

Proof. We first prove that $\dot{\zeta}_r$ is well-defined. Assume $\pi_{\lambda\mu}(u) = 0$ where $u \in \mathbf{U}(\infty)$ and $\lambda, \mu \in \Lambda(\infty, r)$. By the definition of ${}_\lambda\mathbf{U}(\infty)_\mu$ we can write

$$u = \sum_{\mathbf{j}\in\mathbb{Z}^{\infty}} (K^{\mathbf{j}} - \upsilon^{\lambda\cdot\mathbf{j}})u'_{\mathbf{j}} + \sum_{\mathbf{j}\in\mathbb{Z}^{\infty}} u''_{\mathbf{j}}(K^{\mathbf{j}} - \upsilon^{\mu\cdot\mathbf{j}})$$

where $u'_{\mathbf{j}}, u''_{\mathbf{j}} \in \mathbf{U}(\infty)$. By 5.2 we have $[\mathrm{diag}(\lambda)](\Bbbk^{\mathbf{j}} - \upsilon^{\lambda\cdot\mathbf{j}}) = 0 = (\Bbbk^{\mathbf{j}} - \upsilon^{\mu\cdot\mathbf{j}})[\mathrm{diag}(\mu)]$ for $\mathbf{j} \in \mathbb{Z}^{\infty}$. It follows that $[\mathrm{diag}(\lambda)]\zeta_r(u)[\mathrm{diag}(\mu)] = 0$. Hence $\dot{\zeta}_r$ is well defined.

For convenience, we let $[\mathrm{diag}(\lambda)] = 0 \in \mathbf{U}(\infty, r)$ for $\lambda \notin \Lambda(\infty, r)$. Let $t \in \mathbf{U}(\infty)(\lambda' - \mu')$ and $s \in \mathbf{U}(\infty)(\lambda'' - \mu'')$ where $\lambda', \mu', \lambda'', \mu'' \in \mathbb{Z}^{\infty}$ such that $\lambda' - \mu', \lambda'' - \mu'' \in \mathbb{Z}\Pi(\infty)$. If $\mu' \neq \lambda''$ then by the definition of $\dot{\zeta}_r$ we have $\dot{\zeta}_r(\pi_{\lambda'\mu'}(t)\pi_{\lambda''\mu''}(s)) = 0 = \dot{\zeta}_r(\pi_{\lambda'\mu'}(t))\dot{\zeta}_r(\pi_{\lambda''\mu''}(s))$. Now we assume $\mu' = \lambda''$. Then by 7.5 we have $\dot{\zeta}_r(\pi_{\lambda'\mu'}(t))\dot{\zeta}_r(\pi_{\mu'\mu''}(s)) = [\mathrm{diag}(\lambda')]\zeta_r(t)[\mathrm{diag}(\mu')]\zeta_r(s)[\mathrm{diag}(\mu'')] = [\mathrm{diag}(\lambda')]\zeta_r(t)\zeta_r(s)[\mathrm{diag}(\mu'')] = [\mathrm{diag}(\lambda')]\zeta_r(ts)[\mathrm{diag}(\mu'')] = \dot{\zeta}_r(\pi_{\lambda'\mu''}(ts)) = \dot{\zeta}_r(\pi_{\lambda'\mu'}(t)\pi_{\mu'\mu''}(s))$. The result follows. □

Proposition 8.6. *The map $\dot{\zeta}_r$ satisfies the following property*:

$$\dot{\zeta}_r(u_1u_2u_3) = \zeta_r(u_1)\dot{\zeta}_r(u_2)\zeta_r(u_3)$$

where $u_1, u_3 \in \mathbf{U}(\infty)$ and $u_2 \in \dot{\mathbf{U}}(\infty)$.

Proof. We assume $u' \in \mathbf{U}(\infty)(\nu')$, $u'' \in \mathbf{U}(\infty)(\nu'')$ and $u \in \mathbf{U}(\infty)(\lambda-\mu)$ where $\nu',\nu'',\lambda-\mu \in \mathbb{Z}\Pi(\infty)$ and $\lambda,\mu \in \mathbb{Z}^\infty$. By 8.4 we have

$$\dot\zeta_r(u'\pi_{\lambda\mu}(u)u'') = \dot\zeta_r(\pi_{\lambda+\nu',\mu-\nu''}(u'uu'')) = \begin{cases}\zeta_r(u')\zeta_r(u)\zeta_r(u'')[\mathrm{diag}(\mu-\nu'')] & \text{if } \mu-\nu'' \in \Lambda(\infty,r);\\ 0 & \text{otherwise.}\end{cases}$$

Hence by 7.5 we have $\zeta_r(u')\dot\zeta_r(\pi_{\lambda\mu}(u))\zeta_r(u'') = \zeta_r(u')\zeta_r(u)[\mathrm{diag}(\mu)]\zeta_r(u'') = \dot\zeta_r(u'\pi_{\lambda\mu}(u)u'')$. □

Proposition 8.7. *Let $A \in \widetilde{\Xi}(\infty)$. Then we have*

$$\dot\zeta_r([A]) = \begin{cases}[A] & \text{if } A \in \Xi(\infty,r);\\ 0 & \text{otherwise.}\end{cases}$$

In particular we have $\dot\zeta_r(\boldsymbol{\mathcal{K}}(\infty)) = \dot\zeta_r(\dot{\mathbf{U}}(\infty)) = \boldsymbol{\mathcal{K}}(\infty,r)$.

Proof. Let $\lambda = ro(A)$ and $\mu = co(A)$. If either $\lambda \notin \Lambda(\infty,r)$ or $\mu \notin \Lambda(\infty,r)$, then we have $\dot\zeta_r([A]) = \dot\zeta_r([\mathrm{diag}(\lambda)]A^\pm(\mathbf{0})[\mathrm{diag}(\mu)]) = 0$. Now we assume $\lambda,\mu \in \Lambda(\infty,r)$. Then we have

$$\begin{aligned}\dot\zeta_r([A]) &= \dot\zeta_r([\mathrm{diag}(\lambda)]A^\pm(\mathbf{0})[\mathrm{diag}(\mu)])\\ &= [\mathrm{diag}(\lambda)]\zeta_r(A^\pm(\mathbf{0}))[\mathrm{diag}(\mu)]\\ &= [\mathrm{diag}(\lambda)]A^\pm(\mathbf{0},r)[\mathrm{diag}(\mu)]\\ &= \begin{cases}[A] & \text{if } A \in \Xi(\infty,r);\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

The result follows. □

Remark 8.8. The natural linear map from $\boldsymbol{\mathcal{K}}(\infty,r)$ to $\boldsymbol{\mathcal{K}}(\infty)$ by sending $[A]$ to $[A]$ for $A \in \Xi(\infty,r)$ is not algebra homomorphism. For example, in the algebra $\boldsymbol{\mathcal{K}}(\infty,r)$ we have $[E_{1,2}]\cdot[E_{2,1}] = [E_{1,1}]$. However we have $[E_{1,2}]\cdot[E_{2,1}] = [E_{1,1}] + [A]$, where $A = \left(\begin{smallmatrix}0 & 1\\ 1 & -1\end{smallmatrix}\right)$ in the algebra $\boldsymbol{\mathcal{K}}(\infty)$.

By 1.1 we may construct the completion algebra $\widehat{\boldsymbol{\mathcal{K}}}^\dagger(\infty)$ of $\boldsymbol{\mathcal{K}}(\infty)$ such that $\widehat{\boldsymbol{\mathcal{K}}}(\infty)$ becomes an subalgebra of $\widehat{\boldsymbol{\mathcal{K}}}^\dagger(\infty)$.

Theorem 8.9. *There is an algebra epimorphism $\tilde\zeta_r$ from $\widehat{\boldsymbol{\mathcal{K}}}^\dagger(\infty)$ to $\boldsymbol{\mathcal{S}}(\infty,r)$ by sending $\sum_{A\in\widetilde{\Xi}(\infty)}\beta_A[A]$ to $\sum_{A\in\Xi(\infty,r)}\beta_A[A]$. Moreover we have $\tilde\zeta_r(\widehat{\boldsymbol{\mathcal{K}}}(\infty)) = \widehat{\boldsymbol{\mathcal{K}}}(\infty,r)$ and $\tilde\zeta_r|_{\mathbf{U}(\infty)} = \zeta_r$.*

Proof. By 8.7 there is an algebra homomorphism $\tilde\zeta_r$ from $\widehat{\boldsymbol{\mathcal{K}}}^\dagger(\infty)$ to $\boldsymbol{\mathcal{S}}(\infty,r)$ by sending $\sum_{A\in\widetilde{\Xi}(\infty)}\beta_A[A]$ to $\sum_{A\in\widetilde{\Xi}(\infty)}\beta_A\dot\zeta_r([A]) = \sum_{A\in\Xi(\infty,r)}\beta_A[A]$. It is clear we have $\tilde\zeta_r(\widehat{\boldsymbol{\mathcal{K}}}(\infty)) = \widehat{\boldsymbol{\mathcal{K}}}(\infty,r)$. By 7.2 the map $\tilde\zeta_r$ is surjective. By 4.3 and 8.7 we have $\zeta_r(A(\mathbf{j})) = A(\mathbf{j},r) = \tilde\zeta_r(A(\mathbf{j}))$ for $A \in \Xi^\pm(\infty)$ and $\mathbf{j} \in \mathbb{Z}^\infty$. Hence we have $\tilde\zeta_r|_{\mathbf{U}(\infty)} = \zeta_r$. □

Remark 8.10. By [1, 3.10] there is an antiautomorphism τ_r on $\boldsymbol{\mathcal{K}}(\infty,r)$ defined by $\tau_r([A]) = [A^T]$ for $A \in \Xi(\infty,r)$. This induces, by 3.4, an antiautomorphism τ on $\boldsymbol{\mathcal{K}}(\infty)$ defined by $\tau([A]) = [A^T]$ for $A \in \widetilde{\Xi}(\infty)$. Hence, we have ${}^\dagger\widehat{\boldsymbol{\mathcal{K}}}(\infty) \cong (\widehat{\boldsymbol{\mathcal{K}}}^\dagger(\infty))^{op}$ and ${}^\dagger\widehat{\boldsymbol{\mathcal{K}}}(\infty,r) \cong (\widehat{\boldsymbol{\mathcal{K}}}^\dagger(\infty,r))^{op}$.

9. Highest weight representations of $\mathbf{U}(\infty)$

In the rest of the paper, we will discuss the representation theory of $\mathbf{U}(\infty)$. Unlike the case of quantum $\mathfrak{gl}_n$, there are two types of representations for $\mathbf{U}(\infty)$. The first type consists of highest weight representations, while the second type consists of 'polynomial' representations arising from

representations of infinite q-Schur algebras. In this section, we first discuss the highest weight representations of $\mathbf{U}(\infty)$, following the approach used in [24].

Let

$$X(\infty)=\{\lambda=(\lambda_i)_{i\in\mathbb{Z}}\mid\lambda_i\in\mathbb{Z}\},\qquad\text{and}$$
$$X^+(\infty)=\{\lambda\in X(\infty)\mid\lambda_i\geqslant\lambda_{i+1},\ \forall i\in\mathbb{Z}\}\tag{9.0.1}$$

be the sets of weights and dominant weights, respectively. Both $\mathbb{Z}^\infty$ and $\mathbb{N}^\infty$ are subsets of $X(\infty)$. Recall from (4.0.3) that, for $i\in\mathbb{Z}$, $\boldsymbol{e}_i=(\cdots,0,\underset{i}{1},0\cdots)\in\mathbb{Z}^\infty$ and $\alpha_i=\boldsymbol{e}_i-\boldsymbol{e}_{i+1}\in\mathbb{Z}^\infty$. Let $\Pi(\infty)=\{\alpha_i\mid i\in\mathbb{Z}\}$. We introduce a partial ordering $\leqslant_{\rm wt}$ on $X(\infty)$ by setting $\mu\leqslant_{\rm wt}\lambda$ if $\lambda-\mu\in\mathbb{Z}^+\Pi(\infty)$. For $\alpha=\sum_{i\in\mathbb{Z}}k_i\alpha_i\in\mathbb{Z}\Pi(\infty)$ the number $\mathrm{ht}(\alpha):=\sum_{i\in\mathbb{Z}}k_i$ is called the height of α.

We first discuss the 'standard' representation theory of $\mathbf{U}(\infty)$ which covers the category $\mathcal{C}$ of weight modules, the category[4] $\mathcal{C}^{int}$of integrable weight modules, the category $\mathcal{C}^{hi}$ and the category $\mathcal{O}$. All these categories are full subcategories of the category $\mathbf{U}(\infty)$-**Mod** of $\mathbf{U}(\infty)$-modules.

Let M be a $\mathbf{U}(\infty)$-module. For $\lambda\in X(\infty)$, let $M_\lambda=\{x\in M\mid K_ix=v^{\lambda_i}x\text{ for }i\in\mathbb{Z}\}$. If $M_\lambda\neq0$, then λ is called a weight of M, M_λ is called a weight space and a nonzero vector in M_λ is called a weight vector of M. Let $\mathrm{wt}(M)=\{\lambda\in X(\infty)\mid M_\lambda\neq0\}$ denote the set of weights of M. It is clear we have the following lemma.

Lemma 9.1. *Let M be a $\mathbf{U}(\infty)$-module. Then $E_iM_\lambda\subseteq M_{\lambda+\alpha_i}$ and $F_iM_\lambda\subseteq M_{\lambda-\alpha_i}$ for $i\in\mathbb{Z}$.*

A $\mathbf{U}(\infty)$-module M is called a *weight module*, if $M=\oplus_{\lambda\in X(\infty)}M_\lambda$. Let $\mathcal{C}$ denote the full subcategory of $\mathbf{U}(\infty)$-**Mod** consisting of all weight modules. It is clear every submodule of a weight module and the quotient of a weight module are weight modules. Hence $\mathcal{C}$ is an abelian category.

An object $M\in\mathcal{C}$ is said to be integrable if, for any $x\in M$ and any $i\in\mathbb{Z}$, there exist $n_0\geqslant1$ such that $E_i^nx=F_i^nx=0$ for any $n\geqslant n_0$. Let $\mathcal{C}^{int}$ be the full subcategory of $\mathcal{C}$ whose objects are integrable $\mathbf{U}(\infty)$-modules.

Let $\mathcal{C}^{hi}$ be the full subcategory of $\mathcal{C}$ whose objects are the $\mathbf{U}(\infty)$-module M with the following property: for any $x\in M$ there exist $n_0\geqslant1$ such that $u.x=0$ where u is the monomial in the E_i's and $\deg(u)\geqslant n_0$.

For $\lambda\in X(\infty)$ set $(-\infty,\lambda]=\{\mu\in X(\infty)\mid\mu\leqslant_{\rm wt}\lambda\}$. The category $\mathcal{O}$ is defined as follows. Its objects are $\mathbf{U}(\infty)$-modules M which are weight modules with finite dimensional weight spaces and such that there exists a finite number of elements $\lambda^{(1)},\cdots,\lambda^{(s)}\in X(\infty)$ such that $\mathrm{wt}(M)\subseteq\bigcup_{i=1}^s(-\infty,\lambda^{(i)}]$.

Proposition 9.2. *The category $\mathcal{O}$ is a full subcategory of $\mathcal{C}^{hi}$.*

Proof. Let M be a object of $\mathcal{O}$. Then there exist $\lambda^{(i)}(1\leqslant i\leqslant s)$ such that $\mathrm{wt}(M)\subseteq\bigcup_{i=1}^s(-\infty,\lambda^{(i)}]$. Let $x_\mu\neq0\in M_\mu$. Let $n_\mu=\max\{\mathrm{ht}(\lambda^{(i)}-\mu)\mid\lambda^{(i)}\geqslant_{\rm wt}\mu,1\leqslant i\leqslant s\}$. We claim that if $k\geqslant n_\mu+1$, then for any $i_1,\cdots,i_k\in\mathbb{Z}$ we have $\mu+\alpha_{i_1}+\alpha_{i_2}+\cdots+\alpha_{i_k}\notin\mathrm{wt}(M)$. Indeed, if there exists $1\leqslant j\leqslant s$ such that $\mu+\alpha_{i_1}+\alpha_{i_2}+\cdots+\alpha_{i_k}\leqslant\lambda^{(j)}$, then $\mathrm{ht}(\lambda^{(j)}-\mu)\geqslant k\geqslant n_\mu+1$, contrary to the definition of n_μ. Thus, for any $k\geqslant n_{\mu+1}$ and any $i_1,\cdots,i_k\in\mathbb{Z}$, we have $E_{i_1}\cdots E_{i_k}x_\mu\in M_{\mu+\alpha_{i_1}+\cdots+\alpha_{i_k}}=0$. Hence, M is a object of $\mathcal{C}^{hi}$. □

A $\mathbf{U}(\infty)$-module M is called a *highest weight module* with highest weight $\lambda\in X(\infty)$ if there exists a nonzero vector $x_0\in M_\lambda$ such that $E_ix_0=0$, $K_ix_0=v^{\lambda_i}x_0$ for all $i\in\mathbb{Z}$ and $\mathbf{U}(\infty)x_0=M$. The vector x_0 is called a highest weight vector. By a standard argument (see, e.g., [21]) we have the following lemma.

[4] This category is denoted as $\mathcal{C}'$ in [24].

Lemma 9.3. *Let M be a highest weight $\mathbf{U}(\infty)$-module with highest weight λ. Then $M = \bigoplus_{\mu\leqslant\lambda} M_\mu$ and $\dim M_\lambda = 1$. Moreover M contains a unique maximal submodule.*

For $\lambda \in X(\infty)$, let

$$M(\lambda) = \mathbf{U}(\infty)/(\sum_{i\in\mathbb{Z}} \mathbf{U}(\infty)E_i + \sum_{i\in\mathbb{Z}} \mathbf{U}(\infty)(K_i - v^{\lambda_i})) = \mathbf{U}^-(\infty)1_\lambda,$$

where 1_λ is the image of 1 in $M(\lambda)$. The module $M(\lambda)$ is called a *Verma module*. By the above lemma, we know that $M(\lambda)$ has a unique irreducible quotient module $L(\lambda)$. It is clear, for $\lambda \in X(\infty)$, the modules $M(\lambda)$ and $L(\lambda)$ are all in the category $\mathcal{C}^{hi}$ and the category $\mathcal{O}$.

The next result classifies the irreducible modules in the categories $\mathcal{C}^{hi}$ and $\mathcal{O}$.

Theorem 9.4. (1) *The map $\lambda \to L(\lambda)$ defines a bijection between $X(\infty)$ and the set of isomorphism classes of irreducible $\mathbf{U}(\infty)$-modules in the category $\mathcal{C}^{hi}$.*

(2) *The map $\lambda \to L(\lambda)$ defines a bijection between $X(\infty)$ and the set of isomorphism classes of irreducible $\mathbf{U}(\infty)$-modules in the category $\mathcal{O}$.*

Proof. Let M be a irreducible $\mathbf{U}(\infty)$-module in the category $\mathcal{C}^{hi}$. Choose a nonzero vector $x_\mu \in M_\mu$ for some $\mu \in \mathrm{wt}(M)$. Let

$$n_0 = \max\{\deg(u) \mid \text{there exists some monomial } u \text{ in the } E_i\text{'s such that } u.x_\mu \neq 0\}.$$

Since $M \in \mathcal{C}^{hi}$, $n_0 < \infty$. Let u be the monomial in the E_i's such that $\deg(u) \geqslant n_0$ and $u.x_\mu \neq 0$. Then $u.x_\mu$ is a highest weight vector of M. Let $\lambda \in X(\infty)$ be the weight of $u.x_\mu$. Then there exists a surjective homomorphism from $M(\lambda)$ to M. Since $L(\lambda)$ is a unique irreducible quotient module of $M(\lambda)$ we have $M \cong L(\lambda)$, proving (1). The statement (2) follows from (1) and 9.2. $\square$

Corollary 9.5. *An irreducible module $L(\lambda)$ in $\mathcal{C}^{hi}$ is integrable if and only if $\lambda \in X^+(\infty)$.*

Proof. If $\lambda \in X^+(\infty)$, then the module $L(\lambda)$ is a homomorphic image of the module $M(\lambda)/I(\lambda)$, where $I(\lambda) = \sum_{i\in\mathbb{Z}} \mathbf{U}^-(\infty)F_i^{(\lambda_i-\lambda_{i+1}+1)}1_\lambda$. Note that $I(\lambda)$ is a proper submodule of $M(\lambda)$. By [24, 3.5.3], $M(\lambda)/I(\lambda)$ is integrable. Hence, $L(\lambda)$ is integrable. Conversely, let $L(\lambda)$ be a irreducible module in the category $\mathcal{C}^{hi}$, and assume that $L(\lambda)$ is integrable. Then, there exists $x_0 \in M_\lambda$, $x_0 \neq 0$ and $E_i x_0 = 0$ for all $i \in \mathbb{Z}$. By [24, (3.5.8)], $\lambda \in X^+(\infty)$. $\square$

Thus, using 9.2, we have the following classification theorem.

Theorem 9.6. (1) *The map $\lambda \to L(\lambda)$ defines a bijection between $X^+(\infty)$ and the set of isomorphism classes of irreducible $\mathbf{U}(\infty)$-modules in the category $\mathcal{C}^{int} \cap \mathcal{C}^{hi}$.*

(2) *The map $\lambda \to L(\lambda)$ defines a bijection between $X^+(\infty)$ and the set of isomorphism classes of irreducible $\mathbf{U}(\infty)$-modules in the category $\mathcal{C}^{int} \cap \mathcal{O}$.*

The integrable module $L(\lambda)$ has actually structure similar to finite dimensional irreducible $\mathbf{U}([-n,n])$-modules. Recall from [16, 5.15] that, if $\mu \in X^+([-n,n])$, then the irreducible $\mathbf{U}([-n,n])$-module $L(\mu) \cong \mathbf{U}([-n,n])/I_n(\mu)$ where

$$I_n(\mu) = \big(\sum_{-n\leqslant i<n} (\mathbf{U}([-n,n])E_i + \mathbf{U}([-n,n])F_i^{\mu_i-\mu_{i+1}+1}) + \sum_{-n\leqslant j\leqslant n} \mathbf{U}([-n,n])(K_j - v^{\mu_j})\big).$$

We want to prove that a similar isomorphism holds for $\mathbf{U}(\infty)$.

For any given $\lambda \in X^+(\infty)$, let $\widetilde{L(\lambda)} = M(\lambda)/I(\lambda)$, and let $\lambda_{[-n,n]} = (\lambda_i)_{-n\leqslant i\leqslant n} \in X^+([-n,n])$. Let $\pi_{n,\lambda} : \mathbf{U}([-n,n]) \to L(\lambda_{[-n,n]})$ be the map sending u to $u\bar{1}_\lambda$, where $\bar{1}_\lambda$ is the image of 1_λ. Since $I_n(\lambda_{[-n,n]}) \subset I_{n+1}(\lambda_{[-n-1,n+1]})$, there is a natural $\mathbf{U}([-n,n])$-module homomorphism $\iota_{n,\lambda} : L(\lambda_{[-n,n]}) \to L(\lambda_{[-n-1,n+1]})$ by sending $\pi_{n,\lambda}(u)$ to $\pi_{n+1,\lambda}(u)$ for all $u \in \mathbf{U}([-n,n])$, which

is compatible with the inclusion $\mathbf{U}([-n,n]) \hookrightarrow \mathbf{U}([-n-1,n+1])$. Thus, we obtain a direct system $\{L(\lambda_{[-n,n]})\}_{n\geqslant 1}$ whose direct limit $\varinjlim_n L(\lambda_{[-n,n]})$ is naturally a $\mathbf{U}(\infty)$-module . Further, since $\iota_{n,\lambda}(\pi_{n,\lambda}(1)) = \pi_{n+1,\lambda}(1) \neq 0$, the map $\iota_{n,\lambda}$ is injective. So we may identify $L(\lambda_{[-n,n]})$ as a $\mathbf{U}([-n,n])$-submodule of $L(\lambda_{[-n-1,n+1]})$. Hence $\varinjlim_n L(\lambda_{[-n,n]}) = \bigcup_{n\geqslant 1} L(\lambda_{[-n,n]})$.

Theorem 9.7. *We have, for $\lambda \in X^+(\infty)$,*

$$L(\lambda) \cong \varinjlim_n L(\lambda_{[-n,n]}) \cong \widetilde{L(\lambda)}.$$

Proof. Let W be a $\mathbf{U}(\infty)$-submodule of $\varinjlim_n L(\lambda_{[-n,n]})$. Since $L(\lambda_{[-n,n]})$ is irreducible $\mathbf{U}([-n,n])$-module for $n \geqslant 1$ we have $W \cap L(\lambda_{[-n,n]}) = 0$ or $W \cap L(\lambda_{[-n,n]}) = L(\lambda_{[-n,n]})$. If there exist $n_0 \geqslant 1$ such that $W \cap L(\lambda_{[-n,n]}) \neq 0$ then $W \cap L(\lambda_{[-n,n]}) \neq 0$ for $n \geqslant n_0$. Hence $W \cap L(\lambda_{[-n,n]}) = L(\lambda_{[-n,n]})$ for $n \geqslant n_0$. It follows that

$$W = \bigcup_{n\geqslant 1}(W \cap L(\lambda_{[-n,n]})) = \bigcup_{n\geqslant n_0} = (W \cap L(\lambda_{[-n,n]})) = \bigcup_{n\geqslant n_0} L(\lambda_{[-n,n]}) = \varinjlim_n L(\lambda_{[-n,n]}).$$

On the other hand, if $W \cap L(\lambda_{[-n,n]}) = 0$ for $n \geqslant 1$ then $W = 0$. Hence $\varinjlim_n L(\lambda_{[-n,n]})$ is an irreducible $\mathbf{U}(\infty)$-module. Let $x_0 := \pi_{1,\lambda}(1) \in \varinjlim_n L(\lambda_{[-n,n]})$. Then $x_0 = \pi_{n,\lambda}(1)$ for any $n \geqslant 1$. Hence for $n \geqslant 1$ and $1 \leqslant i < n$, $1 \leqslant j \leqslant n$ we have $E_i x_0 = \pi_{n,\lambda}(E_i) = 0$ and $K_j x_0 = \pi_{n,\lambda}(K_j) = \upsilon^{\lambda_i} x_0$. Hence $\varinjlim_n L(\lambda_{[-n,n]})$ is a irreducible highest weight module with highest weight λ. It follows that $L(\lambda) \cong \varinjlim_n L(\lambda_{[-n,n]})$. For any $n \geqslant 1$ there is a natural $\mathbf{U}([-n,n])$-module homomorphism f_n from $L(\lambda_{[-n,n]})$ to $\widetilde{L(\lambda)}$ by sending $\pi_{n,\lambda}(u)$ to $\bar{u}$ for $u \in \mathbf{U}([-n,n])$. The maps f_n ($n \geqslant 1$) induce a surjective $\mathbf{U}(\infty)$-homomorphism f from $\varinjlim_n L(\lambda_{[-n,n]})$ to $\widetilde{L(\lambda)}$. Since $\varinjlim_n L(\lambda_{[-n,n]})$ is irreducible $\mathbf{U}(\infty)$-module and $\mathrm{Im}(f) = \widetilde{L(\lambda)} \neq 0$, f has to be an isomorphism. $\square$

Remark 9.8. By [11, 4.4] and [4, §6] we can use the PBW basis $\{A(\mathbf{0}) \mid A \in \Xi^-(\infty)\}$ of $\mathbf{U}^-(\infty)$ to define the canonical basis $\{\mathfrak{c}_A \mid A \in \Xi^-(\infty)\}$ of $\mathbf{U}^-(\infty)$. For $\lambda \in X^+(\infty)$ let x_λ be the highest weight vector of $L(\lambda)$. By [23, 8.10] and 9.7, one can easily show that the set $\{\mathfrak{c}_A x_\lambda \mid A \in \Xi^-(\infty)\}\backslash\{0\}$ forms a $\mathbb{Q}(\upsilon)$ basis of $L(\lambda)$.

Finally, it should be interesting to point out that there are not many *finite dimensional* weight $\mathbf{U}(\infty)$-modules.

For $n \geqslant 0$, let $J(n)$ be the two sided ideal of $\mathbf{U}(\infty)$ generated by E_i, F_i, $i \in (-\infty,-n) \cup [n,\infty)$.

Lemma 9.9. (1) *For $n \geqslant 0$, we have $J(n) = J(0)$.*

(2) *Let M be a finite dimensional $\mathbf{U}(\infty)$-module in the category $\mathcal{C}$. Then $E_i M = F_i M = 0$ for all $i \in \mathbb{Z}$ and $\mathrm{wt}(M) \subseteq \{\lambda \in X(\infty) \mid \lambda = k\mathbf{1}\}$, where $\mathbf{1} = (\cdots,1,1,\cdots,1,\cdots) \in X(\infty)$.*

Proof. Since $E_n, F_n \in J(n)$, by 2.1(e) we have $\widetilde{K}_n - \widetilde{K}_n^{-1} \in J(n)$. Hence $K_n^2 - K_{n+1}^2 \in J(n)$. By 2.1(b) we have $K_n^2 E_{n-1} = \upsilon^{-2} E_{n-1} K_n^2$ and $K_{n+1}^2 E_{n-1} = E_{n-1} K_{n+1}^2$. Hence $\upsilon^{-2} E_{n-1} K_n^2 - E_{n-1} K_{n+1}^2 = (K_n^2 - K_{n+1}^2) E_n \in J(n)$. It follows that $(\upsilon^{-2} - 1) E_{n-1} K_n^2 = (\upsilon^{-2} E_{n-1} K_n^2 - E_{n-1} K_{n+1}^2) - E_{n-1}(K_n^2 - K_{n+1}^2) \in J(n)$. Hence $E_{n-1} \in J(n)$. It is similar we can show $F_{n-1}, E_{-n}, F_{-n} \in J(n)$. Hence, we have $J(n) = J(n-1)$ for $n \geqslant 1$, and (1) follows.

Since M is finite dimensional weight module, we see that $\mathrm{wt}(M)$ is a finite set. Assume $\mathrm{wt}(M) = \{\lambda^{(i)} \mid 1 \leqslant i \leqslant s\}$. By 9.1 for $1 \leqslant i \leqslant s$ there exist n_i such that $E_j M_{\lambda^{(i)}} = F_j M_{\lambda^{(i)}} = 0$ for $j \notin [-n_i, n_i)$. Let $n_0 = \max\{n_i \mid 1 \leqslant i \leqslant s\}$. then $E_i M = F_i M = 0$ for $i \notin [-n_0, n_0)$. Hence by (1) we have $E_i M = F_i M = 0$ for all $i \in \mathbb{Z}$. Let $\lambda \in \mathrm{wt}(M)$ and x_0 be a nonzero vector in M_λ. By 2.1(e) we have $\widetilde{K}_i^2 x_0 = v^{2\lambda_i - 2\lambda_{i+1}} x_0 = 0$. Hence, $\lambda_i = \lambda_{i+1}$ for all $i \in \mathbb{Z}$, proving (2). □

Let $\mathcal{C}^{fd}$ be the category of finite dimensional weight $\mathbf{U}(\infty)$-module. Now by the above result we have the following classification of finite dimensional $\mathbf{U}(\infty)$-modules.

Theorem 9.10. *The modules $L(k\mathbf{1})$ $(k \in \mathbb{Z})$ are all non-isomorphic finite dimensional irreducible $\mathbf{U}(\infty)$-module in the category $\mathcal{C}^{fd}$ and $\dim(L(k\mathbf{1})) = 1$. Moreover every finite dimensional $\mathbf{U}(\infty)$-module in the category $\mathcal{C}$ is complete reducible.*

10. Representations of $\boldsymbol{\mathcal{S}}(\infty, r)$

In this section, we investigate the weight modules for $\boldsymbol{\mathcal{S}}(\infty, r)$ and classify those irreducible ones.

Recall from §2 and §7 the basis $\{T_w\}_{w \in \mathfrak{S}_r}$ for the Hecke algebra $\boldsymbol{\mathcal{H}}$ and the basis $\{\phi_{\lambda\mu}^w\}$ for the algebra $\boldsymbol{\mathcal{K}}(\infty, r)$ given in (7.2.2). Recall also the various notations for the idempotents

$$\mathtt{k}_\lambda = [\mathrm{diag}(\lambda)] = \phi_{\lambda\lambda}^1, \qquad \lambda \in \Lambda(\infty, r).$$

Let

$$\varpi = (\varpi_i)_{i \in \mathbb{Z}} \in \Lambda(\infty, r), \quad \text{where } \varpi_i = \begin{cases} 1, & \text{if } 1 \leqslant i \leqslant r; \\ 0, & \text{otherwise.} \end{cases} \tag{10.0.1}$$

We first observe the following which is clear from the definition of the †-completion $\widehat{\boldsymbol{\mathcal{K}}}^\dagger(\infty, r)$ and the identification $\boldsymbol{\mathcal{S}}(\infty, r) = \widehat{\boldsymbol{\mathcal{K}}}^\dagger(\infty, r)$ (see 7.3).

Lemma 10.1. (1) *Let $\lambda, \mu \in \Lambda(\infty, r)$. Then $\boldsymbol{\mathcal{S}}(\infty, r)\mathtt{k}_\lambda = \mathbf{U}(\infty, r)\mathtt{k}_\lambda = \boldsymbol{\mathcal{K}}(\infty, r)\mathtt{k}_\lambda$, and $\mathtt{k}_\lambda \boldsymbol{\mathcal{S}}(\infty, r)\mathtt{k}_\mu = \mathrm{Hom}_{\boldsymbol{\mathcal{H}}}(x_\mu \boldsymbol{\mathcal{H}}, x_\lambda \boldsymbol{\mathcal{H}})$.*

(2) *The Hecke algebra $\boldsymbol{\mathcal{H}}$ is isomorphic to $\mathtt{k}_\varpi \boldsymbol{\mathcal{S}}(\infty, r)\mathtt{k}_\varpi$ by sending T_u to $\phi_{\varpi,\varpi}^u$ $(u \in \mathfrak{S}_r)$.*

(3) *Let η be a consecutive segment of $\mathbb{Z}$. Then, whenever $|\eta| \geqslant r$, $\boldsymbol{\mathcal{H}}$ is isomorphic to a (centralizer) subalgebra $e\boldsymbol{\mathcal{S}}(\eta, r)e$ of $\boldsymbol{\mathcal{S}}(\eta, r)$ for some idempotent e.*

An $\boldsymbol{\mathcal{S}}(\infty, r)$-module M is called a *weight module* if $M = \oplus_{\lambda \in \Lambda(\infty, r)} \mathtt{k}_\lambda M$. Clearly, 10.1(1) implies that $\boldsymbol{\mathcal{K}}(\infty, r)$ is a weight $\boldsymbol{\mathcal{S}}(\infty, r)$-module. Let $\boldsymbol{\mathcal{S}}(\infty, r)$-**mod** be the category of weight $\boldsymbol{\mathcal{S}}(\infty, r)$-modules. Since all $\mathtt{k}_\lambda \in \boldsymbol{\mathcal{K}}(\infty, r)$, we can define similarly the categories $\boldsymbol{\mathcal{K}}(\infty, r)$-**mod** and $\mathbf{U}(\infty, r)$-**mod**.

Proposition 10.2. (1) *Up to category isomorphism, the categories $\boldsymbol{\mathcal{K}}(\infty, r)$-**mod**, $\mathbf{U}(\infty, r)$-**mod** and $\boldsymbol{\mathcal{S}}(\infty, r)$-**mod** are all the same.*

(2) *The category $\boldsymbol{\mathcal{S}}(\infty, r)$-**mod** is a full subcategory of $\mathcal{C}$.*

Proof. The statement (1) follows easily from part (1) of the lemma above. We now prove (2). Using the homomorphism $\zeta_r : \mathbf{U}(\infty) \to \boldsymbol{\mathcal{S}}(\infty, r)$, every $\boldsymbol{\mathcal{S}}(\infty, r)$-module M is a $\mathbf{U}(\infty)$-module. It suffices to prove that $\mathtt{k}_\lambda M = M_\lambda$ for all $\lambda \in \Lambda(\infty, r)$. If $x \in M_\lambda$, then

$$\mathtt{k}_\lambda x = \prod_{i \in \mathbb{Z}} \left(\begin{bmatrix} \mathtt{k}_i; 0 \\ \lambda_i \end{bmatrix} x \right) = \prod_{i \in \mathbb{Z}} \begin{bmatrix} \lambda_i \\ \lambda_i \end{bmatrix} x = x.$$

Hence, $M_\lambda \subseteq \mathtt{k}_\lambda M$. Conversely, the inclusion $\mathtt{k}_\lambda M_\lambda \subseteq M_\lambda$ follows from 5.1(3). □

Let $\boldsymbol{\Omega}(\infty,r) := \boldsymbol{\mathcal{S}}(\infty,r)\mathtt{k}_\varpi$. If we identify $\boldsymbol{\mathcal{H}}$ with the subalgebra $\mathtt{k}_\varpi\boldsymbol{\mathcal{S}}(\infty,r)\mathtt{k}_\varpi$ of $\boldsymbol{\mathcal{S}}(\infty,r)$ via the isomorphism given in 10.1(2), then $\boldsymbol{\Omega}(\infty,r) = \bigoplus_{\mu\in\Lambda(\infty,r)} \phi^1_{\mu\varpi}\boldsymbol{\mathcal{H}}$. Hence, $\boldsymbol{\Omega}(\infty,r)$ is an $(\boldsymbol{\mathcal{S}}(\infty,r),\boldsymbol{\mathcal{H}})$-bimodule.

Proposition 10.3. *The evaluation map*

$$\mathrm{Ev} : \boldsymbol{\Omega}(\infty,r) \longrightarrow \boldsymbol{\Omega}_\infty^{\otimes r},\ f \mapsto f(T_1)$$

defines an $(\boldsymbol{\mathcal{S}}(\infty,r),\boldsymbol{\mathcal{H}})$-bimodule isomorphism. Moreover, we have

$$\boldsymbol{\mathcal{H}} \cong \mathrm{End}_{\boldsymbol{\mathcal{S}}(\infty,r)}(\boldsymbol{\Omega}_\infty^{\otimes r}) \cong \mathrm{End}_{\mathbf{U}(\infty)}(\boldsymbol{\Omega}_\infty^{\otimes r}).$$

Proof. Since the set $\{\phi^d_{\mu\varpi} \mid \mu\in\Lambda(\infty,r), d\in\mathfrak{D}_\mu\}$ forms a basis for $\boldsymbol{\Omega}(\infty,r)$, and the set $\{\phi^d_{\mu\varpi}(T_1) \mid \mu\in\Lambda(\infty,r), d\in\mathfrak{D}_\mu\}$ forms a basis for $\boldsymbol{\Omega}_\infty^{\otimes r}$. So the assertion follows easily. □

For $\lambda\in\Lambda(\infty,r)$, let $\lambda^t = (\lambda^t_i)_{i\in\mathbb{Z}} \in \Lambda(\infty,r)$, where $\lambda^t_i = \#\{j\in\mathbb{Z}|\lambda_j \geqslant i\}$ for $i\geqslant 1$ and $\lambda^t_i = 0$ for $i\leqslant 0$. Let w_λ be the unique element in $\mathfrak{D}_{\lambda,\lambda^t}$ such that $w_\lambda^{-1}\mathfrak{S}w_\lambda\cap\mathfrak{S}_{\lambda^t} = \{1\}$, and let $z_\lambda = \phi^1_{\lambda\varpi}T_{w_\lambda}y_{\lambda^t}$, where $y_{\lambda^t} = \sum_{w\in\mathfrak{S}_{\lambda^t}}(-\upsilon^2)^{-l(w)}T_w$.

For any $\mu\in\Lambda([-n,n],r)$, let

$$S^\mu = z_\mu\boldsymbol{\mathcal{H}}, \quad \text{and} \quad W(\infty,\mu) = \boldsymbol{\mathcal{S}}(\infty,r)z_\mu \tag{10.3.1}$$

be the *Specht module* of $\boldsymbol{\mathcal{H}}$ and the *Weyl module* of $\boldsymbol{\mathcal{S}}(\infty,r)$, respectively. Note that, if $\mu\in\Lambda([-n,n],r)$ and $n\geqslant r$, then $\varpi\in\Lambda([-n,n],r)$. So

$$W([-n,n],\mu) := \boldsymbol{\mathcal{S}}([-n,n],r)z_\mu$$

is well-defined. This is a Weyl module of $\boldsymbol{\mathcal{S}}([-n,n],r)$. Since $\boldsymbol{\mathcal{S}}([-n,n],r)$ is a finite dimensional semisimple algebra, by [5, 4.6], $W([-n,n],\mu)$ is an irreducible $\boldsymbol{\mathcal{S}}([-n,n],r)$-module. Note that $z_\mu = \mathtt{k}_\mu z_\mu$. So 10.1 implies $W(\infty,\mu) = \boldsymbol{\mathcal{S}}(\infty,r)\mathtt{k}_\mu z_\mu = \boldsymbol{\mathcal{K}}(\infty,r)\mathtt{k}_\mu z_\mu = \boldsymbol{\mathcal{K}}(\infty,r)z_\mu$. Hence, for any $\mu\in\Lambda(\infty,r)$, we obtain

$$W(\infty,\mu) = \bigcup_{n\geqslant r} W([-n,n],\mu) \cong \varinjlim_n W([-n,n],\mu). \tag{10.3.2}$$

Let $\Lambda^+(r)$ be the set of all partitions of r. We will regard $\Lambda^+(r)$ as the subset

$$\{\lambda\in\Lambda(\infty,r) \mid \lambda_i\geqslant\lambda_{i+1}\forall i\geqslant 1, \lambda_i = 0\,\forall i\leqslant 0\}$$

of $\Lambda(\infty,r)$. Define a map from $\Lambda(\infty,r)$ to $\Lambda^+(r)$ by sending λ to λ^+ where λ^+ is the unique element in $\Lambda^+(r)$ obtained by reordering the parts of λ.

Proposition 10.4. (1) *For $\mu\in\Lambda(\infty,r)$ we have*

$$W(\infty,\mu) \cong \mathrm{Hom}_{\boldsymbol{\mathcal{H}}}(S^\mu, \boldsymbol{\Omega}_\infty^{\otimes r}) \cong \bigoplus_{\lambda\in\Lambda(\infty,r)} \mathrm{Hom}(S^\mu, x_\lambda\boldsymbol{\mathcal{H}}).$$

(2) *Let $\mu\in\Lambda(\infty,r)$. Then the module $W(\infty,\mu)$ is an irreducible $\boldsymbol{\mathcal{S}}(\infty,r)$-module. Moreover, we have $W(\infty,\mu)\cong W(\infty,\mu^+)$.*

Proof. The bimodule isomorphism Ev given in 10.3 induces $\boldsymbol{\mathcal{S}}(\infty,r)$-module isomorphism

$$\mathrm{Hom}_{\boldsymbol{\mathcal{H}}}(S^\mu, \boldsymbol{\Omega}_\infty^{\otimes r}) \cong \mathrm{Hom}_{\boldsymbol{\mathcal{H}}}(S^\mu, \boldsymbol{\Omega}(\infty,r)) \cong \bigoplus_{\lambda\in\Lambda(\infty,r)} \mathrm{Hom}(S^\mu, \phi^1_{\lambda\varpi}\boldsymbol{\mathcal{H}}).$$

Now 7.1 implies

$$\mathrm{Hom}_{\boldsymbol{\mathcal{H}}}(S^\mu, \boldsymbol{\Omega}_\infty^{\otimes r}) \cong \varinjlim_n \mathrm{Hom}_{\boldsymbol{\mathcal{H}}}(S^\mu, \boldsymbol{\Omega}_{[-n,n]}^{\otimes r}).$$

Choose m_0 such that $\mu \in \Lambda([-m_0, m_0], r)$ and let $r_0 = \max\{r, m_0\}$. Then, for any $n \geqslant r_0$, we have, by [5, 8.1,8.7], isomorphisms $\mathrm{Hom}_{\boldsymbol{\mathcal{H}}}(S^\mu, \boldsymbol{\Omega}_{[-n,n]}^{\otimes r}) \cong W([-n,n],\mu)$ which are compatible with the direct systems. Now, the first isomorphism in (1) is obtained by taking direct limits.

The assertions in (2) follow easily from (10.3.2) (cf. [5, 3.9]). □

For $\lambda, \mu \in \Lambda(\infty, r)$ we write $\mu \leqslant^+_{\rm wt} \lambda$ if $\mu^+ \leqslant_{\rm wt} \lambda^+$. Note that $\mu^+ \leqslant_{\rm wt} \lambda^+$ is equivalent to the dominance order $\mu^+ \trianglelefteq \lambda^+$ on partitions.

We also recall the notation of Young tableaux. Suppose $\lambda \in \Lambda(\infty, r)$ and $\mu \in \Lambda^+(r)$. A μ-tableau of type λ is a μ-tableau with (possibly) repeated entries, where for each i, the number of entries i is equal to λ_i. We denote the set of μ-tableaux of type λ by $\mathcal{T}(\mu, \lambda)$. For $T \in \mathcal{T}(\mu, \lambda)$, we say that T is row-standard (resp., strictly rwo-standard) if the numbers are weakly increasing (resp., increasing) along each row of T. The column-standard and strictly column-standard tableaux can be defined similarly. A tableau T is semistandard if it is row-standard and strictly column-standard. Let $\mathcal{T}_0(\mu, \lambda)$ denote the set of semistandard μ-tableaux of type λ.

Proposition 10.5. *Let $\lambda \in \Lambda(\infty, r)$ and $\mu \in \Lambda^+(r)$. We have $W(\infty, \mu)_\lambda = \mathrm{Hom}_{\boldsymbol{\mathcal{H}}}(S^\mu, x_\lambda \boldsymbol{\mathcal{H}})$. Hence, $\dim W(\infty, \mu)_\lambda = \#\mathcal{T}_0(\mu, \lambda)$ ([5, 8.7]). In particular, $\dim W(\infty, \mu)_\mu = 1$, and $W(\infty, \mu)_\lambda \neq 0$ implies $\lambda \leqslant^+_{\rm wt} \mu$.*

Proof. Let $f \in \mathrm{Hom}_{\boldsymbol{\mathcal{H}}}(S^\mu, x_\lambda \boldsymbol{\mathcal{H}})$. Assume $f(z_\mu) = x_\lambda h_\lambda$ where $h_\lambda \in \boldsymbol{\mathcal{H}}$. Then we have

$$\begin{aligned}(\mathtt{k}_i f)(z_\lambda h) = \mathtt{k}_i f(z_\lambda h) = \mathtt{k}_i(x_\lambda h_\lambda h) &= \Big(\sum_{\nu \in \Lambda(\infty, r)} \upsilon^{\nu_i} \phi^1_{\nu\nu}\Big)(x_\lambda h_\lambda h) \\ &= \sum_{\nu \in \Lambda(\infty, r)} \upsilon^{\nu_i} \phi^1_{\nu\nu}(x_\lambda h_\lambda h) = \upsilon^{\lambda_i} x_\lambda h_\lambda h = \upsilon^{\lambda_i} f(z_\lambda h),\end{aligned}$$

where $i \in \mathbb{Z}$ and $h \in \boldsymbol{\mathcal{H}}$. Hence, $\mathtt{k}_i f = \upsilon^{\lambda_i} f$ for $i \in \mathbb{Z}$. This means $f \in W(\infty, \mu)_\lambda$. By 10.4(1), we obtain $(W(\infty, \mu))_\lambda = \mathrm{Hom}_{\boldsymbol{\mathcal{H}}}(S^\mu, x_\lambda \boldsymbol{\mathcal{H}})$. □

Though the double centralizer property in the Schur-Weyl duality is no longer true in the infinite case, the following decomposition for the tensor space $\boldsymbol{\Omega}_\infty^{\otimes r}$ continue to hold.

Theorem 10.6. *We have the following ($\boldsymbol{\mathcal{S}}(\infty, r)$,$\boldsymbol{\mathcal{H}}$)-bimodule isomorphism*

$$\boldsymbol{\Omega}_\infty^{\otimes r} \cong \bigoplus_{\mu \in \Lambda^+(r)} W(\infty, \mu) \otimes S^\mu.$$

Hence, as a (left) $\boldsymbol{\mathcal{S}}(\infty, r)$-module, $\boldsymbol{\Omega}_\infty^{\otimes r}$ is completely reducible.

Proof. It is well-known that, for all $n \geqslant r$, there are ($\boldsymbol{\mathcal{S}}([-n,n], r)$,$\boldsymbol{\mathcal{H}}$)-bimodule isomorphisms

$$\boldsymbol{\Omega}_{[-n,n]}^{\otimes r} \cong \bigoplus_{\mu \in \Lambda^+(r)} W([-n,n], \mu) \otimes S^\mu.$$

The require isomorphism is obtained by taking direct limits; cf. 7.1 and (10.3.2). □

Corollary 10.7. *For each $\lambda \in \Lambda(\infty, r)$, the $\boldsymbol{\mathcal{S}}(\infty, r)$-module $\boldsymbol{\mathcal{S}}(\infty, r)\mathtt{k}_\lambda$ is completely reducible.*

Proof. Consider the $\boldsymbol{\mathcal{S}}(\infty, r)$-module homomorphism

$$f : \boldsymbol{\mathcal{S}}(\infty, r)\mathtt{k}_\lambda \longrightarrow \boldsymbol{\mathcal{S}}(\infty, r)\mathtt{k}_\varpi \cong \boldsymbol{\Omega}_\infty^{\otimes r}, \;\; u\mathtt{k}_\lambda \mapsto u\mathtt{k}_\lambda \phi^1_{\lambda\varpi} \, \forall u \in \boldsymbol{\mathcal{S}}(\infty, r).$$

Since f is induced from the surjective map $\phi^1_{\lambda\varpi} : \boldsymbol{\mathcal{H}} \to x_\lambda \boldsymbol{\mathcal{H}}$, it follows that f is injective. Hence, $\boldsymbol{\mathcal{S}}(\infty, r)\mathtt{k}_\lambda$ is isomorphic to a submodule of $\boldsymbol{\Omega}_\infty^{\otimes r}$. Hence, the assertion follows from 10.6. □

We are now ready to classify irreducible objects in the category $\boldsymbol{\mathcal{S}}(\infty, r)$-**mod**. We first observe the following.

Theorem 10.8. (1) *The map $\mu \to W(\infty,\mu)$ defines a bijection between $\Lambda^+(r)$ and the set of isomorphism classes of irreducible $\boldsymbol{\mathcal{S}}(\infty,r)$-modules in the category $\boldsymbol{\mathcal{S}}(\infty,r)$-**mod**.*

(2) *Any module in $\boldsymbol{\mathcal{S}}(\infty,r)$-**mod** is completely reducible.*

Proof. By 10.4 and 10.5, the modules $W(\infty,\mu)$ ($\mu\in\Lambda^+(r)$) are irreducible objects in $\boldsymbol{\mathcal{S}}(\infty,r)$-**mod**. Let M be an irreducible $\boldsymbol{\mathcal{S}}(\infty,r)$-module in $\boldsymbol{\mathcal{S}}(\infty,r)$-**mod**. Let $\mu\in\mathrm{wt}(M)$ and x_0 is a nonzero vector in M_μ. Then we have a surjective homomorphism f from $\boldsymbol{\mathcal{S}}(\infty,r)$ to M by sending u to ux_0 for $u\in\boldsymbol{\mathcal{S}}(\infty,r)$. By the proof of 10.2(2), we have $x_0=\mathtt{k}_\mu x_0$, and so $f(\boldsymbol{\mathcal{S}}(\infty,r)\mathtt{k}_\mu)=\boldsymbol{\mathcal{S}}(\infty,r)f(\mathtt{k}_\mu)=\boldsymbol{\mathcal{S}}(\infty,r)\mathtt{k}_\mu x_0=\boldsymbol{\mathcal{S}}(\infty,r)x_0=M$. Hence, M is a quotient module of $\boldsymbol{\mathcal{S}}(\infty,r)\mathtt{k}_\mu$. By 10.7, M is isomorphic to an irreducible component of $\boldsymbol{\mathcal{S}}(\infty,r)\mathtt{k}_\mu$, and hence, to an irreducible component of $\boldsymbol{\Omega}_\infty^{\otimes r}$. Now, 10.6 implies that M is isomorphic to $W(\infty,\mu)$ for some $\mu\in\Lambda^+(r)$, proving (1).

Let N be an arbitrary module in $\boldsymbol{\mathcal{S}}(\infty,r)$-**mod**. Let $0\neq x\in N$. Since $N\in\boldsymbol{\mathcal{S}}(\infty,r)$-**mod**, we have $\boldsymbol{\mathcal{S}}(\infty,r)x=\boldsymbol{\mathcal{K}}(\infty,r)x$ is a quotient module of $\boldsymbol{\mathcal{K}}(\infty,r)$. But, by 10.1,

$$\boldsymbol{\mathcal{K}}(\infty,r)=\bigoplus_{\lambda\in\Lambda(\infty,r)}\boldsymbol{\mathcal{K}}(\infty,r)\mathtt{k}_\lambda=\bigoplus_{\lambda\in\Lambda(\infty,r)}\boldsymbol{\mathcal{S}}(\infty,r)\mathtt{k}_\lambda.$$

Hence, 10.7 implies that the $\boldsymbol{\mathcal{S}}(\infty,r)$-module $\boldsymbol{\mathcal{K}}(\infty,r)$ is completely reducible, and so $\boldsymbol{\mathcal{S}}(\infty,r)x$ is completely reducible. Consequently, N is completely reducible. □

Remark 10.9. Following the construction in [8], we can use the PBW type basis

$$\{[A]=\upsilon^{-d_A}\phi^d_{\lambda\mu}\mid A=\jmath(\lambda,d,\mu),(\lambda,d,\mu)\in\mathfrak{D}(\infty,r)\}$$

for $\boldsymbol{\mathcal{K}}(\infty,r)$ to define the canonical basis $\{\theta^d_{\lambda\nu}\mid(\lambda,d,\mu)\in\mathfrak{D}(\infty,r)\}$ of $\boldsymbol{\mathcal{K}}(\infty,r)$. By [7, 5.3] one can easily show that, for each $\mu\in\Lambda^+(r)$, the set $\{\theta^d_{\lambda\nu}z_\mu\mid(\lambda,d,\nu)\in\mathfrak{D}(\infty,r)\}\backslash\{0\}$ forms a $\mathbb{Q}(\upsilon)$-basis for $W(\infty,\mu)$. This basis is called the *canonical basis* of $W(\infty,\mu)$.

11. Polynomial representations of $\mathbf{U}(\infty)$

We are now ready to classify all irreducible polynomials representations of $\mathbf{U}(\infty)$. Let $\mathcal{C}^{pol}$ be the full subcategory of $\mathcal{C}$ consisting of weight $\mathbf{U}(\infty)$-modules M such that $\mathrm{wt}(M)\subseteq\mathbb{N}^{\infty}$. We call the objects in $\mathcal{C}^{pol}$ *polynomial representations*. Let r be a positive integer and let $\mathcal{C}_r$ be the category of $\mathbf{U}(\infty,r)$-modules which are weight $\mathbf{U}(\infty)$-modules via the homomorphism $\zeta_r:\mathbf{U}(\infty)\to\mathbf{U}(\infty,r)$.

Proposition 11.1. (1) *The category $\mathcal{C}_r$ is a full subcategory of $\mathcal{C}^{pol}\cap\mathcal{C}^{int}$.*

(2) *For each $r\in\mathbb{N}$, there is a surjective homomorphism from $\mathbf{U}(\infty,r+1)$ to $\mathbf{U}(\infty,r)$ by sending the generators $\mathtt{e}_i,\mathtt{f}_i,\mathtt{k}_i$ for $\mathbf{U}(\infty,r+1)$ to the generators $\mathtt{e}_i,\mathtt{f}_i,\mathtt{k}_i$ for $\mathbf{U}(\infty,r)$, respectively. Hence, $\mathcal{C}_r$ is a full subcategory of $\mathcal{C}_{r+1}$.*

Proof. Let M be a $\mathbf{U}(\infty,r)$-module in the category $\mathcal{C}_r$. As a $\mathbf{U}(\infty)$-module, E_i,K_i,F_i act on M as the action of $\mathtt{e}_i,\mathtt{k}_i,\mathtt{f}_i$, respectively. But, by 5.4, we have $\mathtt{e}_i^n=\mathtt{f}_i^n=0$ for all $n\geqslant r+1$. Hence, M is an integrable $\mathbf{U}(\infty)$-module.

We now prove $\mathrm{wt}(M)\subseteq\mathbb{Z}^{\infty}$. Suppose this is not the case. Then there exists $\lambda\in\mathrm{wt}(M)$ and $\lambda\notin\mathbb{Z}^{\infty}$. Thus, λ has an infinite support. Choose $r+1$ integers $i_1<i_2<\cdots<i_{r+1}$ such that $\lambda_{i_1}\neq0,\cdots,\lambda_{i_{r+1}}\neq0$ and $\mathbf{t}\in\Lambda(\infty,r)$ such that $t_i=1$ for $i\in\{i_1,\cdots,i_{r+1}\}$ and $t_i=0$ for all other i. Let u_λ be a nonzero vector in M_λ. Then

$$0\neq(\upsilon^{\lambda_{i_1}}-1)\cdots(\upsilon^{\lambda_{i_{r+1}}}-1)u_\lambda=\prod_{i\in\mathbb{Z}}[\upsilon^{\lambda_i};t_i]^!u_\lambda=\prod_{i\in\mathbb{Z}}[K_i;t_i]^!u_\lambda=\prod_{i\in\mathbb{Z}}[\mathtt{k}_i;t_i]^!u_\lambda\overset{6.1}{=}0u_\lambda=0,$$

a contradiction. Hence, $\mathrm{wt}(M)\subseteq\mathbb{Z}^{\infty}$. On the other hand, by 6.1 again, we have $[\mathtt{k}_i;r+1]^!u_\lambda=(\upsilon^{\lambda_i}-1)\cdots(\upsilon^{\lambda_i}-\upsilon^r)u_\lambda=0$ for $i\in\mathbb{Z}$, forcing $0\leqslant\lambda_i\leqslant r$ for any $i\in\mathbb{Z}$. Hence, $\lambda\in\mathbb{N}^{\infty}$, proving (1).

Let J_r be the ideal of $\mathbf{U}(\infty)$ generated by $\prod_{i\in\mathbb{Z}}[K_i;t_i]^!$, where $\mathbf{t}=(t_i)_{i\in\mathbb{Z}}\in\Lambda(\infty,r+1)$. By 6.6 we have $\mathbf{U}(\infty,r)$ is isomorphic to $\mathbf{U}(\infty)/J_r$. Let $\mathbf{t}\in\Lambda(\infty,r+2)$. Choose $i_0\in\mathbb{Z}$ such that $t_{i_0}>0$. Then $\prod_{i\in\mathbb{Z}}[K_i;t_i]^!=(K_{i_0}-v^{t_{i_0}-1})(\prod_{i\neq i_0}[K_i;t_i]^!)[K_{i_0};t_{i_0}-1]^!\in J_r$. Hence, $J_{r+1}\subseteq J_r$, and (2) follows. $\square$

Lemma 11.2. *Let M be a irreducible $\mathbf{U}(\infty)$-module in the category $\mathcal{C}^{pol}$. Then $M\in\mathbf{U}(\infty,r)$-**mod** for some $r\geqslant 0$.*

Proof. Let $\lambda\in\mathrm{wt}(M)$ and u_λ be a nonzero vector in M_λ. Then $M=\mathbf{U}(\infty)u_\lambda$ since M is irreducible. Hence, by 9.1, we know $\sigma(\mu)=\sigma(\lambda)$ for any $\mu\in\mathrm{wt}(M)$. Let $r=\sigma(\lambda)$. Since $\mathrm{wt}(M)\subseteq\mathbb{N}^\infty$ we have $\mathrm{wt}(M)\subseteq\Lambda(\infty,r)$. Thus, for any $\mathbf{t}\in\Lambda(\infty,r+1)$, there exist $i_0\in\mathbb{Z}$ such that $t_{i_0}>\lambda_{i_0}\geqslant 0$. Hence, $[v^{\lambda_{i_0}};t_{i_0}]^!=0$, and $\prod_{i\in\mathbb{Z}}[\mathtt{k}_i;t_i]^!u_\lambda=\prod_{i\in\mathbb{Z}}[v^{\lambda_i};t_i]^!u_\lambda=0$. This shows that M is a $\mathbf{U}(\infty,r)$-module (by 6.6). Since $M\in\mathcal{C}$, it follows that $M\in\mathbf{U}(\infty,r)$-**mod**. $\square$

Theorem 11.3. (1) *For $r\in\mathbb{N}$ the modules $W(\infty,\mu)$ ($\mu\in\Lambda^+(r'),r'\leqslant r$) are all non-isomorphic irreducible $\mathbf{U}(\infty,r)$-modules in the category $\mathcal{C}_r$.*

(2) *The modules $W(\infty,\mu)$ ($\mu\in\Lambda^+(r),r\in\mathbb{N}$) are all non-isomorphic irreducible $\mathbf{U}(\infty)$-modules in the category $\mathcal{C}^{pol}$. Moreover, there is only one irreducible $\mathbf{U}(\infty)$-module, the trivial module $L(0)$, in the category $\mathcal{C}^{pol}\cap\mathcal{C}^{hi}$.*

Proof. By 10.4(2) and 11.1(2) we know $W(\infty,\mu)$, where $\mu\in\Lambda^+(r')$ with $r'\leqslant r$, are irreducible $\mathbf{U}(\infty,r)$-modules in $\mathcal{C}_r$. On the other hand, let M be a irreducible $\mathbf{U}(\infty,r)$-module in the category $\mathcal{C}_r$. By 11.1(1) and 11.2 we have $M\in\mathbf{U}(\infty,r')$-**mod** for some $r'\geqslant 0$. We claim that $r'\leqslant r$. Indeed, suppose the contrary $r'>r$. Then, for any $\lambda\in\mathrm{wt}(M)\subseteq\Lambda(\infty,r')$, there exists $\mu\in\Lambda(\infty,r+1)$ such that $\mu_i\leqslant\lambda_i$ for all $i\in\mathbb{Z}$. Thus, for any nonzero vector x_0 in M_λ,

$$\prod_{i\in\mathbb{Z}}[K_i;\mu_i]^!x_0=\prod_{i\in\mathbb{Z}}[\mathtt{k}_i;\mu_i]^!x_0=\prod_{i\in\mathbb{Z}}[v^{\lambda_i};\mu_i]^!x_0\neq 0$$

for $i\in\mathbb{Z}$. However, since M is a $\mathbf{U}(\infty,r)$-module, the presentation of $\mathbf{U}(\infty,r)$ implies $\prod_{i\in\mathbb{Z}}[K_i;\mu_i]^!x_0=0$, a contradiction. Hence, $r'\leqslant r$. Now 10.8 implies that there exists $\mu\in\Lambda^+(r')$ such that $M\cong W(\infty,\mu)$, proving (1).

The first assertion in (2) follows from 10.8 and 11.2. By 11.1, $W(\infty,\mu)$ is integrable. Since $\mathrm{wt}(W(\infty,\mu))\subseteq\Lambda(\infty,r)$, by the classification of irreducible integral modules in $\mathcal{C}^{hi}$ given in 9.6, we conclude $W(\infty,\mu)\notin\mathcal{C}^{hi}$ for any $r\geqslant 1$ and $\mu\in\Lambda(\infty,r)$, proving the last assertion in (2). $\square$

Remark 11.4. (1) Classification of irreducible integrable modules over a Kac-Moody algebra is an open problem (see [19, Ex. 10.23]). However, irreducible integrable $\mathbf{U}(\infty,r)$-modules can be classified for all $r\in\mathbb{N}$. In fact, by the above result, the modules $W(\infty,\mu)$ ($\mu\in\Lambda^+(r'),r'\leqslant r$) are all irreducible integrable $\mathbf{U}(\infty,r)$-modules for any $r\in\mathbb{N}$.

It is clear that $\boldsymbol{\mathcal{S}}(\infty,r)\cong\prod_{\mu\in\Lambda(\infty,r)}\boldsymbol{\mathcal{S}}(\infty,r)\mathtt{k}_\mu$ as a $\boldsymbol{\mathcal{S}}(\infty,r)$-module. Hence, $\boldsymbol{\mathcal{S}}(\infty,r)\notin\mathcal{C}$ as a $\mathbf{U}(\infty)$-module. Since $\boldsymbol{\mathcal{S}}(\infty,r)\mathtt{k}_\mu$ is a direct sum of finitely many irreducible $\boldsymbol{\mathcal{S}}(\infty,r)$-module in $\boldsymbol{\mathcal{S}}(\infty,r)$-**mod**, $\boldsymbol{\mathcal{S}}(\infty,r)$ is the direct product of irreducible $\boldsymbol{\mathcal{S}}(\infty,r)$-module in $\boldsymbol{\mathcal{S}}(\infty,r)$-**mod**.

(2) It would be interesting to make a comparison between the finite and infinite cases for quantum $\mathfrak{gl}_\eta$. In the finite case, if we only consider finite dimensional representations, then the categories $\mathcal{C}^{hi}$ and $\mathcal{C}^{int}$ are the same as the category $\mathcal{C}^{fd}$, and contain $\boldsymbol{\mathcal{S}}(n,r)$-**mod** as a full subcategory of polynomial representations of degree r which more or less determine $\mathcal{C}^{hi}$ and $\mathcal{C}^{int}$. They are all completely reducible and the irreducible modules in these categories are all indexed by dominant weights. In the infinite case, the situation is completely different. For example, the categories $\mathcal{C}^{hi}\cap\mathcal{C}^{int}$, $\boldsymbol{\mathcal{S}}(\infty,r)$-**mod** and $\mathcal{C}^{fd}$ become three different categories. The complete reducibility continue to hold in the last two categories, but seems not true in the first category. The irreducible

modules in $\mathcal{C}^{hi} \cap \mathcal{C}^{int}$ are also indexed by dominant weights. Moreover, there are only a few irreducible objects in $\mathcal{C}^{fd}$, and there is only one irreducible object in the category $\mathcal{C}^{pol} \cap \mathcal{C}^{hi}$.

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